TMA4250 Spatial Statistics
Exam 21.05.2010
Suggested problem solutions
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Problem 1. Continuous RF

$$\{R(x); x \in D \subset \mathbb{R}^2\} \rightarrow \begin{cases} E\{R(x)\} = \mu_R(x) \\ Var\{R(x)\} = \sigma_R^2(x) \\ Corr\{R(x'), R(x'')\} = \rho_R(x', x'') \end{cases}$$

a)

$[R(x_1), \dots, R(x_n)] \sim$ Gaussian pdf
all conf $(x_1, \dots, x_n)$
all $n > 0$

$$\text{Stationary, isotropic} \Rightarrow \begin{cases} \mu_R(x) = \mu_R \\ \sigma_R^2(x) = \sigma_R^2 \\ \rho_R(x', x'') = \rho_R(\|x' - x''\|) \end{cases}$$

$$S(x) = B^T g(x) + \underbrace{R(x)}_{\text{stat, isotr}}$$

$B \sim \text{Gauss}(b; 0, \sigma_B^2 I)$

$g(x)$ - known

$B, R(x)$ - indep.

b)

$S(x) \sim$ Gauss RF since linear comb of Gauss RV and Gauss RF.

$$E\{S(x)\} = E\{B^T\} g(x) + E\{R(x)\} = \mu_R$$

$$\sigma_S^2(x) = Var\{S(x)\} = g^T(x)\, \sigma_B^2 I\, g(x) + \sigma_R^2$$
$$= \sigma_B^2\, g^T(x) g(x) + \sigma_R^2$$

$c_S(x', x'')$

$$C_S \quad \mathrm{Cov}\{S(x'), S(x'')\} = \sigma_B^2\, g^T(x') g(x'') + \sigma_R^2 \rho_R(\|x' - x''\|)$$

$$\mathrm{corr}\{S(x'), S(x'')\} = \frac{c_S(x', x'')}{\sigma_S(x')\,\sigma_S(x'')}$$

$$s(x^d_S) = (s(x_1), \ldots, s(x_n))^T$$

c)
$$\begin{bmatrix} S(x_0) \\ S(x_1) \\ \vdots \\ S(x_n) \end{bmatrix} \sim \mathrm{Gauss}_{n+}\left(\begin{bmatrix} \mu_R \\ \\ \\ \mu_R \end{bmatrix}, \begin{bmatrix} \sigma_S^2(x_0) & c_S(x_1, x_0) & \\ c_S(x_0, x_1) & \sigma_S^2(x_1) & c_S(x_1, x_n) \\ c_S(x_0, x_n) & c_S(x_1, x_n) & \sigma_S^2(x_n) \end{bmatrix} \right)$$

$$\sim \mathrm{Gauss}_{n+1}\left(\begin{bmatrix} \mu_R \\ \mu_R \end{bmatrix}, \begin{bmatrix} \sigma_S^2(x_0) & \mathbb{c}_{d0} \\ \mathbb{c}_{0d} & \mathbb{C}_{dd} \end{bmatrix} \right)$$

$$[S(x_0) \mid s(x^d)] \sim \mathrm{Gauss}_1(\mu_{0|d}, \sigma_{0|d}^2)$$

$$\mu_{0|d} = \mu_R + c_{0|d}\, \mathbb{C}_{dd}^{-1} (s(x^d) - \mu_R)$$

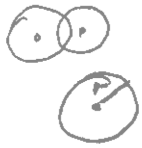
$$\sigma_{0|d}^2 = \sigma_S^2(x_0) - c_{0|d}\, \mathbb{C}_{dd}^{-1}\, c_{0|d}^T$$

$$\mathrm{Prob}\{S(x_0) > s_0 \mid S(x^d) = s(x^d)\} = 1 - \Phi(s_0; \mu_{0|d}, \sigma_{0|d}^2)$$

Stand. Gauss cdf

Problem 2. Event RF $\nearrow$ NB!

$\{X_i\,;\, i=1,\dots,N\,;\, D \subset \mathbb{R}^3\}$



constant circular radius r

a) $1\ \mathrm{d}\ldots\ \mathcal{B} \subset D \qquad |\mathcal{B}| = 1\,\mathrm{dm}^3$

$$N(\mathcal{B}) \Rightarrow \mathrm{Poisson}(n;\lambda|\mathcal{B}|) =$$

$$\mathrm{Prob}\{N(\mathcal{B})=k\} = \frac{[\lambda|\mathcal{B}|]^k}{k!}e^{-\lambda|\mathcal{B}|} = \frac{\lambda^k}{k!}e^{-\lambda}$$

b)

$$E\{\text{weight of } dx \,/\, \text{cheese in } dx\} = \rho\, dx \qquad dx \subset D$$

$$E\{\text{weight of } dx \,/\, \text{hole in } dx\} = 0$$

hence

$$E\{\text{weight of } dx\} = E\{w \text{ of } dx \,|\, \text{ch in } dx\}\cdot \mathrm{Prob}\{\text{ch in } dx\} + \underbrace{E\{w \text{ of } dx \,|\, \text{ho in } dx\}}_{=0}\cdot \mathrm{Prob}\{\text{ho in } dx\}$$

$$= \rho \cdot dx \cdot \mathrm{Prob}\{\text{ch in } dx\}$$

$$= \rho\, dx \cdot \mathrm{Prob}\{dx \text{ dist closest ho-centre} > r\}$$

$$= \rho\, dx \cdot \frac{[\lambda \frac{4}{3}\pi r^3]^0}{0!} e^{-\lambda\frac{4}{3}\pi r^3}$$

$$= \rho e^{-\lambda \frac{4}{3}\pi r^3}\, dx$$

$$E\{\text{weight of } \mathcal{B}\} = \int_{\mathcal{B}} \rho e^{-\lambda\frac{4}{3}\pi r^3}\, dx = \rho e^{-\lambda\frac{4}{3}\pi r^3}|\mathcal{B}|$$

$$= \underline{\rho e^{-\lambda \frac{4}{3}\pi r^3}}$$

c)

$\mathcal{D}_1$ = dist. from arb. loc $x \in D$ to closest ho-centre

= dist from ho-centre to closest ho-centre

↑ Property of Poisson RF

$$\text{Prob}\{\mathcal{D}_1 > d\} = \text{Prob}\{ (d) - \text{empty} \} = \frac{\left[\lambda \frac{4}{3}\pi d^3\right]^0}{0!} e^{-\lambda \frac{4}{3}\pi d^3}$$

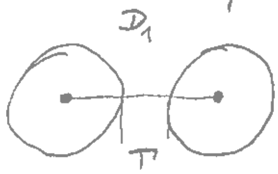
$$= e^{-\lambda \frac{4}{3}\pi d^3} \qquad d > 0$$

$$F_{\mathcal{D}_1}(d) = \text{Prob}\{\mathcal{D}_1 < d\} = 1 - e^{-\lambda \frac{4}{3}\pi d^3} \qquad d > 0$$

$$f_{\mathcal{D}_1}(d) = \frac{dF_{\mathcal{D}_1}}{dd} = 3\lambda \frac{4}{3}\pi d^2 e^{-\lambda \frac{4}{3}\pi d^3} \qquad d > 0$$

T - minimum cheese-thickness / circular hole

$$= \left[\mathcal{D}_1 - 2r \mid \mathcal{D}_1 > 2r\right]$$



Change of variable:

$t = d - 2r \Rightarrow d = t + 2r$

$t > 0 \Rightarrow d > 2r$

$$f_T(t) = f_D(t + 2r \mid d > 2r) \qquad t > 0$$

$$= \frac{f(t+2r)}{1 - F_D(2r)}$$

$$= e^{+\lambda \frac{4}{3}\pi (2r)^3} \cdot 3\lambda \frac{4}{3}\pi (t+2r)^2 e^{-\lambda \frac{4}{3}\pi (t+2r)^3} \; ; t > 0$$

$$= 3\lambda \frac{4}{3}\pi (t+2r)^3 e^{-\lambda \frac{4}{3}\pi \left[(t+2r)^3 - (2r)^3\right]} \qquad t > 0$$

Problem 3 Mosaic Random Field

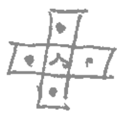
$L : \{L_s ; s \in \mathcal{L}_D\}$ $\quad L_s \in [-1, 1]$ $\quad \mathcal{L}_D$ reg. latt over $D \subset \mathbb{R}^2$

Ising model - par β

$$\text{Prob}\{L = l\} = \text{const} \times \exp\Big\{-\sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v\rangle}} \beta\, l_u l_v\Big\}$$

a) Markov formulation

$$\text{Prob}\{L_s = l_s \mid L_r = l_r ;\ r \in \delta(s)\} = \frac{\exp\Big\{-\sum_{\substack{u \in \mathcal{L}_D \\ u \in \delta(s)}} \beta\, l_s l_u\Big\}}{\sum_{l'_s \in [-1,1]} \exp\Big\{-\sum_{\substack{u \in \mathcal{L}_D \\ u \in \delta(s)}} \beta\, l'_s l_u\Big\}} \quad ; \forall s \in \mathcal{L}_D$$

$\delta(s) =$  $\quad - \quad s = (i, j)$

$\delta(s) = [(i-1, j), (i, j-1), (i+1, j), (j, i+1)]$

b) $$f(d \mid L = l) = \prod_{t \in \mathcal{L}_D} f(d_t \mid L_r = l_r,\ r \in \nu(t))$$

$$\text{Prob}\{L \mid d\} = \text{const} \times f(d \mid L = l)\, \text{Prob}\{L = l\}$$

$$= \text{const} \prod_{t \in \mathcal{L}_D} f(d_t \mid L_r = l_r;\ r \in \nu(t))\, \exp\Big\{-\sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle}} \beta\, l_u l_v\Big\}$$

Markov formulation:

$$\text{Prob}\{L_s = l_s \mid L_{-s} = l_{-s},\ d\}$$

$$= \frac{\text{Prob}\{\overbrace{L_s = l_s,\ L_{-s} = l_{-s}}^{L = l} \mid d\}}{\sum_{l_s' \in [-1,1]} \text{Prob}\{L_s = l_s',\ L_{-s} = l_{-s} \mid d\}}$$

$$= \frac{\text{const} \prod_{\substack{t \in \mathcal{L}_D \\ t \notin \nu(s)}} f(d_t \mid L_r = l_r;\ r \in \nu(t))\, \exp\Big\{-\sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle \\ u,v \neq s}} \beta\, l_u l_v\Big\}}{\text{const} \prod_{\substack{t \in \mathcal{L}_D \\ t \in \nu(s)}} f(d_t \mid L_r = l_r;\ r \in \nu(t))\, \exp\Big\{-\sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle \\ u,v \neq s}} \beta\, l_u l_v\Big\}} = 1$$

$$\frac{\prod_{\substack{t \in \mathcal{L}_D \\ t \in \nu(s)}} f(d_t \mid L_r = l_r;\ r \in \nu(t)) \cdot \exp\Big\{-\sum_{\substack{u \in \mathcal{L}_D \\ u \in \partial(s)}} \beta\, l_s l_u\Big\}}{\sum_{l_s' \in [-1,1]} \prod_{\substack{t \in \mathcal{L}_D \\ t \in \nu(s)}} f(d_t \mid L_r = l_r;\ r \in \nu(t)) \cdot \exp\Big\{-\sum_{\substack{u \in \mathcal{L}_D \\ u \in \partial(s)}} \beta\, l_s' l_u\Big\}}$$

$$= \text{Prob}\{L_s = l_s \mid d,\ L_r = l_r;\ r \in \partial(s) \cup \zeta(s)\}$$

with $\zeta(s)$ – neighborh assoc with clique $\nu(s)$