SUGGESTED SOLUTION

Exam TMA4250 Spatial Statistics

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Suggested solution:

Problem 1 Continuous RF

$\{R(x); x \in D\}$ $\quad E\{R(x)\} = \mu_R \quad Var\{R(x)\} = \sigma_R^2$

$Corr\{R(x''), R(x')\} = \rho_R(\tau)$

a) $\rho_R(\tau)$ need to be a non-negative definite function,

hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \rho_R(\tau_{ij}) \geq 0 \quad \begin{cases} \text{all } \alpha = (\alpha_1, \ldots, \alpha_n) \\ \text{all } n > 1 \\ \tau_{ij} = |x_i - x_j| \end{cases}$$

$$\gamma(\tau) = \frac{1}{2} Var\{R(x') - R(x'')\}$$

$$= \frac{1}{2}\left[2\sigma_R^2 - 2\,Cov\{R(x'), R(x'')\}\right]$$

$$= \sigma_R^2 \left[1 - \rho_R(\tau)\right]$$

$\{S(x) = a + R(x) + U(x); x \in D\}$

indep $R(x), U(x)$

$E\{U(x)\} = 0$

$Var\{U(x)\} = \sigma_U^2$

$Corr\{U(x'), U(x'')\} = I(x' = x'')$

b)

$$\mu_S = a + \mu_R$$

$$\sigma_S^2 = \sigma_R^2 + \sigma_U^2$$

$$C_S(\tau) = \operatorname{Cov}\{S(x'), S(x'')\} = \operatorname{Cov}\{R(x'), R(x'')\} + \sigma_U^2 I(x''=x')$$

$$= C_R(\tau) + \sigma_U^2 I(\tau=0)$$

with

$$C_R(\tau) = \sigma_R^2 \rho_R(\tau)$$

hence

$$\rho_S(\tau) = [\sigma_R^2 + \sigma_U^2]^{-1} [\sigma_R^2 \rho_R(\tau) + \sigma_U^2 I(\tau=0)]$$

Lastly

$$\sigma_{RS}^2 = \sigma_R^2$$

$$C_{RS}(\tau) = \operatorname{Cov}\{R(x'), S(x'')\} = \operatorname{Cov}\{R(x'), R(x'')\} + \operatorname{Cov}\{R(x'), U(x')\}$$

$$= C_R(\tau)$$

hence

$$\rho_{RS}(\tau) = \left[\sigma_R [\sigma_R^2 + \sigma_U^2]^{1/2}\right]^{-1} \sigma_R^2 \rho_R(\tau)$$

$$= [\sigma_R^2 + \sigma_U^2]^{-1/2} \sigma_R \rho_R(\tau)$$

c) $\hat{R}_+ = \sum_{i=1}^{nr} \alpha_i R_i^0 + \sum_{j=1}^{ns} \beta_j S_j^{10}$

Unbiasedness:

$$E\{\hat{R}_+ - R_+\} = 0 \Rightarrow E\{\hat{R}_+\} = E\{R_+\} = \cancel{\mu_R}$$

$$E\{\hat{R}_+\} = \sum_i \alpha_i \cancel{\mu_R} + \sum_j \beta_j \cancel{\mu_S}$$

$$= \left[\sum_i \alpha_i + \sum_j \beta_j\right] \cancel{\mu_R} + a_1 \sum_j \beta_j = \cancel{\mu_R}$$

$$\Downarrow$$

$$\sum_i \alpha_i = 1 \qquad \sum_j \beta_j = 0$$

Prediction variance:

$$Var\{\hat{R}_+ - R_+\} = Var\{R_+\} - 2 Cov\{\hat{R}_+, R_+\} + Var\{\hat{R}_+\}$$

$$= \sigma_R^2 - 2\left[\sum_i \alpha_i C_R(\tau_{i+}) + \sum_j \beta_j C_{RS}(\tau_{j+})\right]$$

$$+ \left[\sum_i \sum_k \alpha_i \alpha_k C_R(\tau_{ik}) + \sum_j \sum_e \beta_j \beta_e C_S(\tau_{je}) + 2 \sum_i \sum_j \alpha_i \beta_j C_{RS}(\tau_{ij})\right]$$

$$= \sigma_R^2 - 2\sigma_R^2 \left[\sum_i \alpha_i \rho_R(\tau_{i+}) + \sum_j \beta_j \rho_R(\tau_{j+})\right]$$

$$+ \left[\sigma_R^2 \sum_i \sum_k \alpha_i \alpha_k \rho_R(\tau_{ik}) + \sigma_R^2 \sum_j \sum_e \beta_j \beta_e \rho_R(\tau_{je}) + \sigma_U^2 \sum_j \beta_j^2 + 2\, \sigma_R^2 \sum_i \sum_j \alpha_i \beta_j \rho_R(\tau_{ij})\right]$$

The weights (α, β) can be defined by:

$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \; Var\{\hat{R}_+ - R_+\}$$

$$\sum_i \alpha = 1 \quad \text{and} \quad \sum_j \beta = 0$$

If observation design is identical:

$$nr = ns \qquad \tau_{i+} = \tau_{j+} \qquad \tau_{ik} = \tau_{jl} = \tau_{ij}$$

and

$$S_i^{\circ} = a + R_i^{\circ} + U_i^{\circ} \; ; \; i = 1, \ldots, ns$$

Then

$$\hat{R}_+ = \sum \alpha_i R_i^{\circ} + \sum \beta_i \left(a + R_i^{\circ} + U_i^{\circ}\right)$$

$$= \sum (\alpha_i + \beta_i) R_i^{\circ} + \sum \beta_i U_i^{\circ}$$

and

$$Var\{\hat{R}_+ - R_+\} = Var\{R_+\} - 2\,Cov\{\hat{R}_+, R_+\} + Var\{\hat{R}_+\}$$

$$= \sigma_R^2 - 2\sigma_R^2 \sum_i (\alpha_i + \beta_i)\rho_R(\tau_{i+})$$

$$+ \sigma_R^2 \sum_i \sum_k (\alpha_i + \beta_i)(\alpha_k + \beta_k)\rho_R(\tau_{ik}) + \sigma_U^2 \sum_i \beta_i^2$$

The solution is:

$$(\hat{\alpha}, \hat{\beta}) = \underset{\substack{\alpha,\beta \\ \sum \alpha_i = 1 \quad \sum_i \beta_i = 0}}{\operatorname{argmin}} Var\{\hat{R}_+ - R_+\}$$

Substitute

$$\left.\begin{aligned} \nu_i &= \alpha_i + \beta_i \\ \beta_i &= \nu_i - \alpha_i \end{aligned}\right\} \sum \nu_i = 1 \quad \Rightarrow \quad \sum_i \beta_i = 0$$

$$(\hat{\alpha}, \hat{\nu}) = \underset{\substack{\alpha,\nu \\ \sum \alpha_i = 1 \quad \sum \nu_i = 1}}{\operatorname{argmin}} \left\{ \sigma_R^2 - 2\sigma_R^2 \sum_i \nu_i \rho_R(\tau_{i+}) + \sigma_R^2 \sum_i \sum_k \nu_i \nu_k \rho_R(\tau_{ik}) + \sigma_\nu^2 \sum_i (\nu_i - \alpha_i)^2 \right\}$$

$$= \underset{\substack{\nu \\ \sum \nu_i = 1 \quad \alpha = \nu}}{\operatorname{argmin}} \left\{ \sigma_R^2 - 2\sigma_R^2 \sum \nu_i \rho_R(\tau_{i+}) + \sigma_R^2 \sum_i \sum_k \nu_i \nu_k \rho_R(\tau_{ik}) \right\}$$

hence

$\hat{\beta}_i = 0 \quad i = 1, \ldots, ns$

$\hat{\alpha}$ – regular Ordinary Kriging weights

If the observation designs are identical the observations S^0 have no influence on the predictor $\hat{R}_+$.

Suggested Solution:

Probleu 2 Event RF

Homog Poisson point RF $\{ \underline{X}_i ; i=1,\dots,N ; \mathcal{D} \in \mathbb{R}^2 \}$

Intensity: $\lambda \, [m^{-2}]$

Area of $\mathcal{D}$: $5 \times 5 \, [m^2]$

a) Probability # points:

$$\text{Prob}\{N = n\} = \frac{[25\lambda]^n}{n!} e^{-25\lambda} \quad ; \; n = 0, 1, 2, \dots$$

- Poisson distributed

Probability 5 points covered in $1\,m^2$; N_Δ:

$$\text{Prob}\{N_\Delta = 5\} = \frac{\lambda^5}{5!} e^{-\lambda}$$

Probability # points in uncovered $24\,m^2$ area $N_{-\Delta}$) given 5 points in covered area:

$$\text{Prob}\{N_{-\Delta} = n \,/\, N_\Delta = 5\} = \text{Prob}\{N_{-\Delta} = n\}$$

indep. of non-overlap area.

$$= \frac{[24\lambda]^n}{n!} e^{-24\lambda} \quad ; \; n = 0, 1, \dots$$

Observation with $(1-p)$-censoring.

b) $$\text{Prob}\{N^\circ = n^\circ \mid N = n\} = \binom{n}{n^\circ} p^{n^\circ} (1-p)^{n-n^\circ} \; ; \; n^\circ = 0, \ldots, n$$

binomial pdf

$$\text{Prob}\{N^\circ = n^\circ\} = \sum_{n=n^\circ}^{\infty} \text{Prob}\{N^\circ = n^\circ \mid N = n\} \, \text{Prob}\{N = n\}$$

$$= \sum_{n=n^\circ}^{\infty} \frac{n!}{(n-n_0)!\, n_0!} \left[\frac{p}{1-p}\right]^{n^\circ} (1-p)^n \frac{(25\lambda)^n}{n!} e^{-25\lambda}$$

$$= \frac{1}{n^\circ!} \left[\frac{p}{1-p}\right]^{n^\circ} e^{-25\lambda} \left[25(1-p)\lambda\right]^{n^\circ} \underbrace{\sum_{n=n^\circ}^{\infty} \frac{\left[25(1-p)\lambda\right]^{n-n^\circ}}{(n-n^\circ)!}}_{= e^{25(1-p)\lambda}}$$

$$= \frac{\left[25 p \lambda\right]^{n^\circ}}{n^\circ!} e^{-25p\lambda} \quad ; \quad n^\circ = 0, 1, \ldots\ldots$$

Pois$(25p\lambda)$

c) Given $N^{\circ} = n^{\circ}$

$$\text{Prob}\{N = n \mid N^{\circ} = n^{\circ}\} = \frac{\text{Prob}\{N = n, N^{\circ} = n^{\circ}\}}{\text{Prob}\{N^{\circ} = n^{\circ}\}}$$

$$= \frac{\text{Prob}\{N^{\circ} = n^{\circ} \mid N = n\}\,\text{Prob}\{N = n\}}{\text{Prob}\{N^{\circ} = n^{\circ}\}}$$

$$= \frac{n!}{(n-n^{\circ})!\,n^{\circ}!}\left[\frac{p}{1-p}\right]^{n^{\circ}}(1-p)^{n}\frac{(25\lambda)^{n}}{n!}e^{-25\lambda}\,n^{\circ}!\,[25p\lambda]^{-n^{\circ}}e^{25p\lambda}$$

$$= \frac{1}{(n-n^{\circ})!}\left[25(1-p)\lambda\right]^{n-n^{\circ}}e^{-25(1-p)\lambda} \quad ; \quad n = n^{\circ}, n^{\circ}+1, \ldots\ldots$$

Independence requires: $N^{-\circ} = N - N^{\circ}$

$$\text{Prob}\{N^{-\circ} = n^{-\circ} \mid N^{\circ} = n^{\circ}\} = \text{Prob}\{N^{-\circ} = n^{-\circ}\}; \; n^{-\circ} = 0, 1, \ldots$$

Note:

$$\text{Prob}\{N^{-\circ} = n^{-\circ}\} = \frac{[25(1-p)\lambda]^{n^{-\circ}}}{n^{-\circ}!}e^{-25(1-p)\lambda}; \; n^{-\circ} = 0, 1, \ldots.$$

from symmetry in reg/non-reg and b)

Moreover:

$$\text{Prob}\{N^{-\circ} = n^{-\circ} \mid N^{\circ} = n^{\circ}\} = \text{Prob}\{N = n^{-\circ} + n^{\circ} \mid N^{\circ} = n^{\circ}\}$$

$$= \frac{[25(1-p)\lambda]^{n^{-\circ}}}{n^{-\circ}!}e^{-25(1-p)\lambda} \qquad n^{-\circ} = 0, 1, \ldots.$$

from above!

Hence - $[N^{\circ}, N - N^{\circ}]$ independent RV!

The predictive power of N° with regard to $N = N^{\circ} + N^{-\circ}$, is only in the first term N°.

Suggested Solution

Problem 3 Mosaic RF

$L : \{ L_x ; x \in \mathcal{L}_D \}$ $L_x \in \Omega_\ell : \{W, G, B\}$ $x \in \mathcal{L}_D$

Gibbs form:

$$\text{Prob}\{ L = \ell ; \beta = (\beta_W, \beta_G, \beta_B) \}$$

$$= \text{const}_\beta \times \exp\Big\{ \sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle}} \sum_{l_u \in \Omega_\ell} \beta_{l_u} I(l_u = l_v) \Big\}$$

a) Markov form:

$$\text{Prob}\{ L_s = \ell_s \mid L_{-s} = \ell_{-s} \}$$

$$= \frac{\text{Prob}\{ L_s = \ell_s , L_{-s} = \ell_{-s} \}}{\text{Prob}\{ L_{-s} = \ell_{-s} \}}$$

$$= \frac{\text{Prob}\{ L = \ell \}}{\sum_{l_s' \in \Omega_\ell} \text{Prob}\{ L_s = \ell_s' , L_{-s} = \ell_{-s} \}}$$

$$= \frac{\text{const}_\beta \times \exp\Big\{ \sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle \\ u,v \neq s}} \sum_{l_u \in \Omega_\ell} \beta_{l_u} I(l_u = l_v) \Big\} \times \exp\Big\{ \sum_{\substack{t \in \mathcal{L}_D \\ \langle s,t \rangle}} \beta_{l_s} I(l_s = l_t) \Big\}}{\text{const}_\beta \times \exp\Big\{ \sum_{\substack{u,v \in \mathcal{L}_D \\ \langle u,v \rangle \\ u,v \neq s}} \sum_{l_u \in \Omega_\ell} \beta_{l_u} I(l_u = l_v) \Big\} \sum_{l_s' \in \Omega_\ell} \exp\Big\{ \sum_{\substack{t \in \mathcal{L}_D \\ \langle s,t \rangle}} \beta_{l_s'} I(l_s' = l_t) \Big\}}$$

=

$$= \frac{\exp\left\{ \sum_{\substack{t \in \mathcal{L}_D \\ \langle s,t \rangle}} \beta_{l_s} I(l_s = l_t) \right\}}{\sum_{l_s' \in \Omega_l} \exp\left\{ \sum_{\substack{t \in \mathcal{L}_D \\ \langle s,t \rangle}} \beta_{l_s'} I(l_s' = l_t) \right\}} \quad ; \forall s \in \mathcal{L}_D$$

Gibbs formulation:

One joint probability model for λ, with a normalizing constant which is very difficult to asses.

Markov formulation:

A set of univariate probability models for each $\lambda_x ; s \in \mathcal{L}_D$, with easily assessible normalizing constants.

b) Observations $l^\circ : \{ l_x ; x \in \mathcal{L}_D \} \to$ estimate $\beta = (\beta_W, \beta_G, \beta_B)$

Ideally we would use maximum likelihood est:

$$\hat{\beta} = \underset{\beta}{\mathrm{argmax}}\ \mathrm{Prob}\{ \lambda = l^\circ ; \beta \}$$

but since the const$_\beta$ is hard to assess, we normally use maximum pseudo-likelihood:

$$\hat{\beta} = \underset{\beta}{\mathrm{argmax}} \left\{ \prod_{s \in \mathcal{L}_D} \mathrm{Prob}\{ \lambda_s = l_s^\circ \mid \lambda_{-s} = l_{-s}^\circ ; \beta \} \right\}$$

This expression does not contain any cumbersome normalizing constants dependent on β.