SUGGESTED SOLUTION EXAM
TMA 4250 SPATIAL STATISTICS
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Ref: H. Omre

PROBLEM 1: CONTINUOUS RANDOM FIELDS

Continuous RF $\{R(x); x \in \underbrace{D}_{[0,10]\times[0,10]} \subset \mathbb{R}^2\}$

$E\{R(x)\} = \mu$ - unknown

$$\left.\begin{aligned} &Var\{R(x)\} = \sigma^2 \\ &corr\{R(x'), R(x)\} = \rho(\underbrace{\tau}_{|x'-x''|}) \end{aligned}\right\} \text{known}$$

$R_D = \underbrace{\frac{1}{|D|}}_{100} \int_D R(u)du$ - spatial average

a) $E\{R_D\} = \frac{1}{|D|}\int_D E\{R(u)\}du = \mu \cdot \frac{|D|}{|D|} = \mu$

$$Var\{R_D\} = E\{R_D^2\} - [E\{R_D\}]^2$$
$$= \frac{1}{|D|^2}\int_D\int_D E\{R(u)R(v)\}du\,dv - \mu^2$$
$$= \frac{1}{|D|^2}\int_D\int_D Cov\{R(u), R(v)\}du\,dv$$
$$= \frac{\sigma^2}{|D|^2}\int_D\int_D \rho(u-v)\,du\,dv$$

$$\mathrm{Cov}\{R(x_0), R_{\mathcal{D}}\} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \mathrm{Cov}\{R(x_0), R(u)\}\,du$$

$$= \frac{\sigma^2}{|\mathcal{D}|}\int_{\mathcal{D}} \rho(x_0 - u)\,du$$

b)

$$\hat{\mu} = \sum_{i=1}^{n} \beta_i R(x_i)$$

unknown weights $\beta = (\beta_1, \dots, \beta_n)$ to be determined.

Unbiasedness:

$$E\{\hat{\mu}\} = \mu \quad \Rightarrow$$

$$\sum_i \beta_i E\{R(x_i)\} = \mu \quad \Rightarrow$$

$$\mu \sum_i \beta_i = \mu \quad \Rightarrow \quad \underline{\sum_i \beta_i = 1}$$

Est. variance under unbiasedness:

$$\mathrm{Var}\{\hat{\mu} - \mu\} = \mathrm{Var}\left\{\sum_i \beta_i [R(x_i) - \mu]\right\}$$

$$= \sum_i \sum_j \beta_i \beta_j \mathrm{Cov}\{R(x_i), R(x_j)\}$$

$$= \sigma^2 \sum_i \sum_j \beta_i \beta_j \rho(x_i - x_j)$$

Best linear unbiased estimator - weights:

$$\hat{\beta} = \mathrm{argmin}_\beta \{\mathrm{Var}_\beta\{\hat{\mu} - \mu\}\}$$

$$= \mathrm{argmin}_\beta \left\{\sum_i \sum_j \beta_i \beta_j \rho(x_i - x_j)\right\}$$

under

$$\sum_i \beta_i = 1.$$

Can use Lagrange minimation to solve this.

c) $\hat{R}_{\mathcal{D}} = \sum_{i=1}^{n} \alpha_i R(x_i)$

unknown weights $\alpha = (\alpha_1, \dots, \alpha_n)$ to be determined

Unbiasedness:

$$E\{\hat{R}_{\mathcal{D}}\} = E\{R_{\mathcal{D}}\} \Rightarrow$$

$$\sum_i \alpha_i \, E\{R(x_i)\} = \mu \Rightarrow$$

$$\mu \sum_i \alpha_i = \mu \Rightarrow \underline{\sum_i \alpha_i = 1}$$

Pred. variance under unbiasedness:

$$Var\{\hat{R}_{\mathcal{D}} - R_{\mathcal{D}}\} = Var\{R_{\mathcal{D}}\} - 2\,Cov\{\hat{R}_{\mathcal{D}}, R_{\mathcal{D}}\} + Var\{\hat{R}_{\mathcal{D}}\}$$

$$= \frac{\sigma^2}{|\mathcal{D}|^2} \int_{\mathcal{D}}\int_{\mathcal{D}} \rho(u-v)\,du\,dv - 2\sum_i \alpha_i \frac{\sigma^2}{|\mathcal{D}|} \int_{\mathcal{D}} \rho(x_i - u)\,du$$

$$+ \sigma^2 \sum_i \sum_j \alpha_i \alpha_j \, \rho(x_i - x_j)$$

Best linear unbiased predictor - weights:

$$\hat{\alpha} = \operatorname{argmin}_{\beta} \{ Var_{\beta}\{\hat{R}_{\mathcal{D}} - R_{\mathcal{D}}\}\}$$

$$= \operatorname{argmin}_{\beta} \Big\{ \sum_i \sum_j \alpha_i \alpha_j \, \rho(x_i - x_j) - \frac{2}{|\mathcal{D}|} \sum_i \alpha_i \int_{\mathcal{D}} \rho(x_i - u)\,du \Big\}$$

under

$$\sum_i \alpha_i = 1$$

Can use Lagrange minimization to solve this.

Note that unbiasedness require normalization of weights in both b) and c) $\sum_i \beta_i = 1$ and $\sum_i \alpha_i = 1$.
In b) the estimator for the model parameter μ only depends on the relative location of $R(x_1), \dots, R(x_n)$ through $\rho(x_i - x_j)$.
In c) the predictor for the average over $\mathcal{D}$ in addition depends on the absolute location of $R(x_1), \dots, R(x_n)$ within $\mathcal{D}$ through $\rho(x_i - u)$ integrated $u \in \mathcal{D}$. More weight to observations centrally located in $\mathcal{D}$.

Hence the estimator $\hat{\mu}$ and predictor $\hat{R}_{\mathcal{D}}$ will be different.

The estimation variance for μ and predictor variance for $R_{\mathcal{D}}$ are of course also different. The former is with respect to a constant μ, while the latter is with respect to a random variable $R_{\mathcal{D}}$.

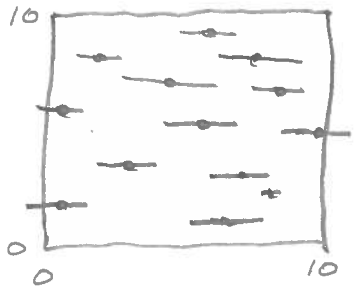
Lastly, if $|\mathcal{D}| \to \infty$, the $\int_{\mathcal{D}} \rho(x-u)\,du < \infty$, hence the estimator and predictor will approach eachother in the limit.

PROBLEM 2: EVENT RANDOM FIELDS

$\{(X_i, l_i);\ i=1,\dots,N;\ D \subset \mathbb{R}^2\}$

l_i ← $f(l)$

X_i ← Poisson RF - para $\lambda>0$



a) N - no. of linesegments
- no of points in D

$$\text{Prob}\{N=n\} = \frac{[\lambda|D|]^n}{n!} e^{-\lambda|D|} = \frac{[100\lambda]^n}{n!} e^{-100\lambda} \quad ; n=0,1,\dots$$

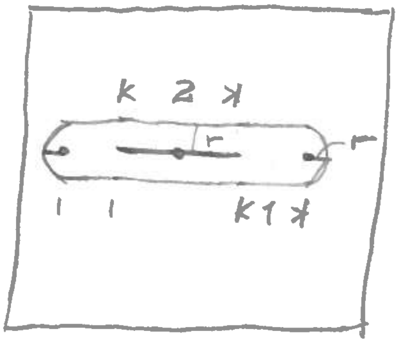
Assume $l_i = l = 2$

N_x - no of linesegments crossing boundary
- no of points closer than 1 to boundary, hence in areas $[0,1]\times[0,10] + [9,10]\times[0,10]$ hence area $10+10=20$

$$\text{Prob}\{N_x=n_x\} = \frac{[20\lambda]^{n_x}}{n_x!} e^{-20\lambda} \quad ; n_x=0,1,\dots$$

$$E\{N_x\} = 20\lambda$$

b)



$\to$ A_r: Area around line-seguent

$(1+2+1)\times 2r + 2\times\frac{1}{2}\pi r^2$

$= 8r+\pi r^2$

R - distance to closest other line-seguent

$F(r) = \text{Prob}\{R<r\}$

$\text{Prob}\{R>r\} = 1-F(r)$

$= \text{Prob}\{\text{no lineseg. centre in } A_r\}$

$= e^{-(8r+\pi r^2)\lambda}$

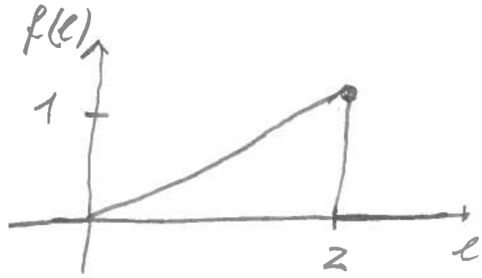
$F(r) = 1-e^{-(8r+\pi r^2)\lambda}$

The pdf for shortest dist.

$f(r) = \dfrac{dF(r)}{dr} = (8+2\pi r)\lambda e^{-(8r+\pi r^2)\lambda}$

c) Assume linesegment length pdf:

$$L \sim f(l) = \begin{cases} \frac{1}{2} l & 0<l<2 \\ 0 & \text{else} \end{cases}$$



Define RV:

$$I_X = \begin{cases} 1 & \text{arbitrary linesegment intersects boundary} \\ 0 & \text{else} \end{cases}$$

For arbitrary linesegment define joint pdf:

$$[L, I_X] \sim \text{Prob}\{I_X = i_X \mid L = l\} f(l)$$

Note

$$\text{Prob}\{I_X = 1 \mid L = l\}$$
$$= \text{Prob}\{X \text{ closer than } l/2 \text{ to boundary}\}$$
$$= \frac{2 \times 10 \times l/2}{100} = \underline{\frac{l}{10}}$$

Hence

$$p_X = \text{Prob}\{I_X = 1\} = \int_{-\infty}^{\infty} \text{Prob}\{I_X = 1 \mid L = l\} f(l)\, dl$$
$$= \frac{1}{20}\int_0^2 l^2\, dl = \frac{1}{60}\Big|_0^2 l^3 = \frac{8}{60} = \underline{\underline{\frac{2}{15}}}$$

Let

N_X - no of linesegments in D that intersects boundary

$$\text{Prob}\{N_X = n_X \mid N = n\} = \binom{n}{n_X} p_X^{n_X}(1-p_X)^{n-n_X} \; ; n_X = 0, 1, \ldots$$

Consequently, the pdf of no. of linesegments intersecting the boundary is:

$$\text{Prob}\{N_x = n_x\} = \sum_{n=0}^{\infty} \text{Prob}\{N_x = n_x \mid N = n\}\,\text{Prob}\{N = n\}$$

$$= \sum_n \binom{n}{n_x} p_x^{n_x} (1-p_x)^{n-n_x} \frac{[100\lambda]^n}{n!} e^{-100\lambda}$$

$$= \frac{[100 p_x \lambda]^{n_x}}{n_x!} e^{-100 p_x \lambda}$$

$$= \frac{\left[\frac{40}{3}\lambda\right]^{n_x}}{n_x!} e^{-\frac{40}{3}\lambda} \quad ; \quad n_x = 0, 1, 2, \ldots$$

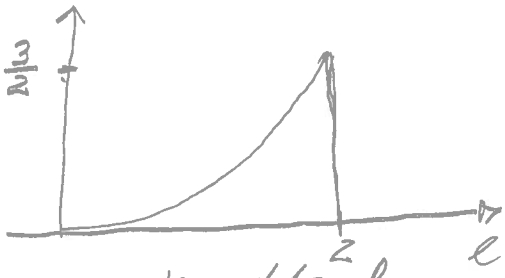
NOTE: This corresponds to thinning by p_x!

Pdf of length of linesegment that intersects boundary:

$$f(\ell \mid i_x = 1) = \left[\text{Prob}\{I_x = 1\}\right]^{-1} \text{Prob}\{I_x = 1 \mid L = \ell\} \times f(\ell)$$

$$= \begin{cases} \left[\frac{2}{15}\right]^{-1} \frac{\ell}{10} \frac{1}{2}\ell & ; \quad 0 < \ell < 2 \\ 0 & \quad \text{else} \end{cases}$$

$$= \begin{cases} \frac{15}{40}\ell^2 & ; \quad 0 < \ell < 2 \\ 0 & \quad \text{else} \end{cases}$$



Note: The length pdf of linesegments that intersects the boundary is different from the length pdf of an arbitrary linesegment. The former tends to be longer than the latter.

PROBLEM 3: MOSAIC RANDOM FIELDS

Consider $L: \{L_x ; x \in \mathcal{L}_D\}$ $\quad l_x \in \Omega_l : \{W, B\}$
$\qquad\qquad\qquad\quad \uparrow *n$

Gibbs formulation:

$$I(A) = \begin{cases} 1 & A \text{ true} \\ 0 & \text{else} \end{cases}$$

$$\text{Prob}\{L = l ; \beta = (\beta_W, \beta_B)\}$$
$$= \text{const}_\beta \times \exp\Big\{\beta_W \sum_u I(l_u = W) + \beta_B \sum_u I(l_u = B) + \frac{1}{2} \times \sum_{\langle u,v \rangle} I(l_u = l_v)\Big\}$$

a)

$\beta = (\beta_W, \beta_B)$ are parameters that influences the expected proportions of W-nodes and B-nodes. Exact proportions in the marginal pdf is not easy to determine, however. Note that for $\beta = (0,0)$, the proportion is 50/50 due to symmetry.

$$\text{Prob}\{L = l ; \beta = (\beta_W, \beta_B)\}$$
$$= \text{const}_\beta \times \exp\Big\{\beta_W \sum_u I(l_u = W) + \beta_B \sum_u I(l_u = B) + \frac{1}{2} \sum_{\langle u,v \rangle} I(l_u = l_v)\Big\}$$
$$= \text{const}_\beta \times \exp\Big\{\beta_W \sum_u I(l_u = W) + \beta_B \sum_u [1 - I(l_u = W)] + \frac{1}{2} \sum_{\langle u,v \rangle} I(l_u = l_v)\Big\}$$
$$= \underbrace{\text{const}_\beta \times \exp\{n\beta_B\}}_{\text{const}^*_\beta} \times \exp\Big\{(\beta_W - \beta_B) \sum_u I(l_u = W) + \frac{1}{2} \sum_{\langle u,v \rangle} I(l_u = l_v)\Big\}$$

Since:
$$[\text{const}'_\beta]^{-1} = \sum_{l \in \Omega_l^n} \exp\Big\{(\beta_W - \beta_B) \sum_u I(l_u = W) + \frac{1}{2} \sum_{\langle u,v \rangle} I(l_u = l_v)\Big\}$$
$$= [\text{const}'_{(\beta_W - \beta_B)}]^{-1}$$

Hence $\text{Prob}\{L = l ; \beta = \beta_W - \beta_B\}$!

b) Markov formulation at $x \in \mathcal{L}_D$:

$$\mathrm{Prob}\{L_x = l_x \mid L_{-x} = l_{-x};\, \Delta\beta\}$$

$$= \frac{\mathrm{Prob}\{L = (l_x, l_{-x});\, \Delta\beta\}}{\sum\limits_{l'_x \in \Omega_l} \mathrm{Prob}\{L = (l'_x, l_{-x});\, \Delta\beta\}}$$

$$= \frac{const_{\Delta\beta}\, \exp\{\Delta\beta\, I(l_x = W) + \Delta\beta \sum\limits_{\substack{u \\ u \neq x}} I(l_u = W)\}}{\sum\limits_{l'_x \in \Omega_l} \Big[const_{\Delta\beta}\, \exp\{\Delta\beta\, I(l'_x = W) + \Delta\beta \sum\limits_{\substack{u \\ u \neq x}} I(l_u = W)\}} \times$$

$$\times \frac{\exp\{\frac{1}{2} \sum\limits_{\substack{<u,v> \\ u,v \neq x}} I(l_u = l_v) + \frac{1}{2} \sum\limits_{<x,u>} I(l_x = l_u)\}}{\exp\{\frac{1}{2} \sum\limits_{\substack{<u,v> \\ u,v \neq x}} I(l_u = l_v) + \frac{1}{2} \sum\limits_{<x,u>} I(l'_x = l_u)\} \Big]}$$

$$= \frac{\exp\{\Delta\beta\, I(l_x = W) + \frac{1}{2} \sum\limits_{<x,u>} I(l_x = l_u)\}}{\sum\limits_{l'_x \in \Omega_l} \exp\{\Delta\beta\, I(l'_x = W) + \frac{1}{2} \sum\limits_{<x,u>} I(l'_x = l_u)\}} \quad ; \forall x \in \mathcal{L}_D$$

NOTE:

Gibbs formulation is one n-dim joint pdf with a $const_{\Delta\beta}$ that is not computable

Markov formulation is n one-dim, conditional pdfs with computable constants.