SUGGESTED SOLUTION / H. Omre 14.06.2016

TMA4250 Spatial Statistics

Exam May 27. 2016

Problem 1 - Continuous RF

$\{R(x); x \in \mathcal{D} \subset \mathbb{R}^2\} \rightarrow$ Expectation μ_r

stationary — Variance σ_r^2

Spat corr. func $\rho_r(\tau); \tau \in \mathbb{R}^1$

$\underset{n\times 1}{\mathbb{R}} : \{R(x); x \in \mathcal{L}_\mathcal{D}\}$

lattice over $\mathcal{D}$

a) Constraints on model parameters

$-\infty < \mu_r < \infty$ – arb. real number

$0 < \sigma_r^2 < \infty$ – arb. pos real number

$\rho_r(0) = 1.0$

$\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j \rho_r(\tau_{ij}) \geq 0 \quad ; \tau_{ij} = |x_i - x_j|$

all conf $[x_1, \ldots, x_m]$

all $\alpha = [\alpha_1, \ldots, \alpha_m]$

$\forall m > 1$

pos. def. function

b) $\underset{n\times 1}{\mathbb{R}} \sim N_n(\underset{n\times 1}{\mu_r}, \underset{n\times n}{\Sigma_r})$

$\mu_r = \mu_r \mathbb{1}_n \qquad \mathbb{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\Sigma_r = \sigma_r^2 \Sigma_r^\rho$

$\Sigma_r^\rho = [\rho_r(\tau_{ij})]_{i,j=1,\ldots,n}$

$\tau_{ij} = |x_i - x_j|$

c) $\mathbb{D} = \mathbb{d}$

$$[\mathbb{D} \mid \mathbb{R} = \mathbb{r}] = \underset{m \times n}{H}\mathbb{r} + \bar{U} \qquad \bar{U} \sim N_m(0\overset{\circ}{\mathbb{1}}_m, \sigma^2_{d|r}\mathbb{I}_m)$$

$$\mathbb{I} \; \underset{m \times m}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$$

$$\Rightarrow N_m(H\mathbb{r}, \sigma^2_{d|r}\mathbb{I}_m)$$

Hence:

$\mathbb{R}$

$\mathbb{D} = H\mathbb{R} + \bar{U}$

and

$$\begin{bmatrix} \mathbb{R} \\ \mathbb{D} \end{bmatrix} \Rightarrow N_{n+m}\left(\begin{bmatrix} \mu_r \overset{\circ}{\mathbb{1}}_n \\ H\mu_r \overset{\circ}{\mathbb{1}}_n \end{bmatrix}, \begin{bmatrix} \sigma_r^2 \Sigma_r^\rho & \sigma_r^2 \Sigma_r^\rho H^T \\ H\sigma_r^2 \Sigma_r^\rho & H\sigma_r^2 \Sigma_r^\rho H^T + \sigma^2_{d|r}\mathbb{I}_m \end{bmatrix} \right)$$

$$[\mathbb{R} \mid \mathbb{D} = \mathbb{d}] \Rightarrow N_n(\mu_{r|d}, \Sigma_{r|d})$$

$$\mu_{r|d} = \mu_r \overset{\circ}{\mathbb{1}}_n + \sigma_r^2 \Sigma_r^\rho H^T \left[H\sigma_r^2 \Sigma_r^\rho H^T + \sigma^2_{d|r}\mathbb{I}_m \right]^{-1} (\mathbb{d} - H\mu_r \overset{\circ}{\mathbb{1}}_n)$$

$$\Sigma_{r|d} = \sigma_r^2 \Sigma_r^\rho - \sigma_r^2 \Sigma_r^\rho H^T \left[H\sigma_r^2 \Sigma_r^\rho H^T + \sigma^2_{d|r}\mathbb{I}_m \right]^{-1} H\sigma_r^2 \Sigma_r^\rho$$

For special case $n = m = 1$

$$\mu_{r|d} = \mu_r + \sigma_r^2 h \left[h\sigma_r^2 h + \sigma^2_{d|r} \right]^{-1} (d - h\mu_r)$$

$$\sigma^2_{r|d} = \sigma_r^2 - \sigma_r^2 h \left[h\sigma_r^2 h + \sigma^2_{d|r} \right]^{-1} h\sigma_r^2$$

d)

$$r^* \Rightarrow N_1(\mu_r, \sigma_r^2)$$
$$d^* \Rightarrow N_1(d, \sigma_{d|r}^2)$$
indep

$$r^s = \underset{r}{\operatorname{argmin}} \Big\{ \underbrace{\frac{(d^* - hr)^2}{\sigma_{d|r}^2} + \frac{(r^* - r)^2}{\sigma_r^2}}_{C(r)} \Big\}$$

$$\frac{dC(r)}{dr} = \frac{2(d^* - hr)(-h)}{\sigma_{d|r}^2} + \frac{2(r^* - r)(-1)}{\sigma_r^2} = 0$$

$$\Downarrow$$

$$\sigma_r^2 h^2 r + \sigma_{d|r}^2 r = \sigma_r^2 d^* h + \sigma_{d|r}^2 r^*$$

$$\Downarrow$$

$$r^s = \frac{\sigma_r^2 d^* h + \sigma_{d|r}^2 r^*}{[\sigma_r^2 h^2 + \sigma_{d|r}^2]}$$

Hence r^s lin. comb d^* and r^* $\Rightarrow$ r^s Gauss

Gauss

$$E(r^s) = \frac{\sigma_r^2 E(d^*) h + \sigma_{d|r}^2 E(r^*)}{[\sigma_r^2 h^2 + \sigma_{d|r}^2]}$$

$$= \frac{\sigma_r^2 dh + \sigma_{d|r}^2 \mu_r + [\sigma_r^2 h^2 \mu_r - \sigma_r^2 h^2 \mu_r]}{\sigma_r^2 h^2 + \sigma_{d|r}^2}$$

$$= \mu_r + \frac{\sigma_r^2 h (d - h\mu_r)}{\sigma_r^2 h^2 + \sigma_{d|r}^2} = E(R \mid D = d) \quad \text{ok!}$$

$$Var(r^s) = \frac{\sigma_r^4 h^2 Var(d^*) + \sigma_{d|r}^4 Var(r^*)}{\left[\sigma_r^2 h^2 + \sigma_{d|r}^2\right]^2}$$

$$= \frac{\sigma_r^4 h^2 \sigma_{d|r}^2 + \sigma_{d|r}^4 \sigma_r^2 + \left[\sigma_r^4 h^4 \sigma_r^2 + 2\sigma_r^2 \sigma_{d|r}^2 h^2 \sigma_r^2 - \sigma_r^4 h^4 \sigma_r^2 - 2\sigma_r^2 \sigma_{d|r}^2 h^2 \sigma_r^2\right]}{\sigma_r^4 h^4 + 2\sigma_r^2 \sigma_{d|r}^2 h^2 + \sigma_{d|r}^4}$$

$$= \frac{\left[\sigma_r^4 h^4 + 2\sigma_r^2 \sigma_{d|r}^2 h^2 + \sigma_{d|r}^4\right]\sigma_r^2}{\left[\sigma_r^4 h^4 + 2\sigma_r^2 \sigma_{d|r}^2 h^2 + \sigma_{d|r}^4\right]}$$

$$+ \frac{-\left[\sigma_r^4 h^4 \sigma_r^2 + \sigma_r^4 \sigma_{d|r}^2 h^2\right]}{\left[\sigma_r^2 h^2 + \sigma_{d|r}^2\right]^2}$$

$$= \sigma_r^2 - \frac{\sigma_r^2 h^2 \sigma_r^2 \left[\sigma_r^4 h^2 + \sigma_{d|r}^2\right]}{\left[\sigma_r^2 h^2 + \sigma_{d|r}^2\right]^2}$$

$$= \sigma_r^2 - \frac{\sigma_r^4 h^2}{\left[\sigma_r^2 h^2 + \sigma_{d|r}^2\right]} = Var(R \mid D = d) \quad \text{ok!}$$

Problem 2 Event RF

Homo Poisson RF $\{X_i ; i=1,\dots,N ; D \subset \mathcal{R}^2\}$ – $\lambda \geq 0$

Define $\mathcal{B} \subset D \rightarrow$ Point count $N(\mathcal{B}) \in \{0,1,2,\dots\}$

a) $$\text{Prob}\{N(\mathcal{B})=i\} = \frac{[\lambda|\mathcal{B}|]^i}{i!}\exp\{-\lambda|\mathcal{B}|\} = p_i$$

$$\Downarrow$$

$$\text{Prob}\{N(\mathcal{B})>0\} = 1-\text{Prob}\{N(\mathcal{B})=0\} = \underline{\underline{1-\exp\{-\lambda|\mathcal{B}|\}}}$$

$$\text{Prob}\{N(\mathcal{B})=i \,|\, N(\mathcal{B})>0\} = p_{i|>0}$$

$$= [1-\exp\{-\lambda|\mathcal{B}|\}]^{-1}\,\text{Prob}\{N(\mathcal{B})=i\} \; ; \; i=1,2,\dots$$

hence

$$p_{i|>0} = [1-\exp\{-\lambda|\mathcal{B}|\}]^{-1}\, p_i$$

$$E\{N(\mathcal{B})\,|\,N(\mathcal{B})>0\} = \sum_{i=1}^{\infty} i\, p_{i|>0}$$

$$= [1-\exp\{-\lambda|\mathcal{B}|\}]^{-1}\sum_{i=1}^{\infty} i\, p_i$$

$$= [1-\exp\{-\lambda|\mathcal{B}|\}]^{-1}\underbrace{\sum_{i=0}^{\infty} i\, p_i}_{=\lambda|\mathcal{B}| = E\{N(\mathcal{B})\}}$$

$$= \underline{\underline{[1-\exp\{-\lambda|\mathcal{B}|\}]^{-1}\,\lambda|\mathcal{B}|}}$$

$$\operatorname{Var}\{N(B) \mid N(B) > 0\}$$

$$= E\{N^2(B) \mid N(B) > 0\} - \left[E\{N(B) \mid N(B) > 0\}\right]^2$$

$$= \left[1 - \exp\{-\lambda |B|\}\right]^{-1} \sum_{i=1}^{\infty} i^2 p_i - [\cdot]^2$$

$$= \left[1 - \exp\{-\lambda |B|\}\right]^{-1} \underbrace{\sum_{i=0}^{\infty} i^2 p_i}_{E\{N^2(B)\} = \operatorname{Var}\{N(B)\} + [E\{N(B)\}]^2} - [\cdot]^2$$

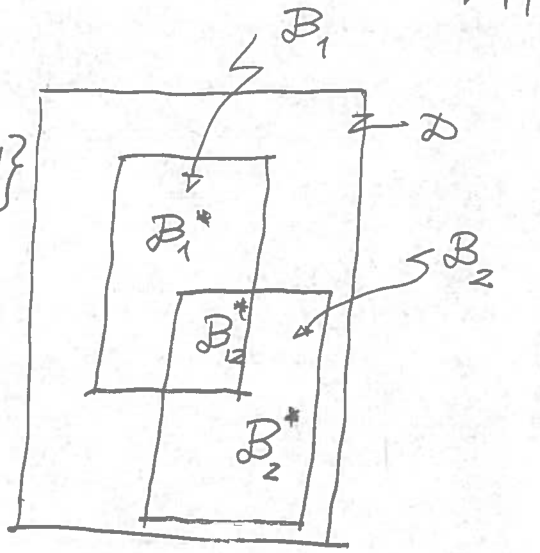
$$\lambda |B| + [\lambda |B|]^2$$

$$= \left[1 - \exp\{-\lambda |B|\}\right]^{-1} \left[\lambda |B| + [\lambda |B|]^2\right]$$

$$- \left[1 - \exp\{-\lambda |B|\}\right]^{-2} [\lambda |B|]^2$$

$$= \left[1 - \exp\{-\lambda |B|\}\right]^{-1} \times \lambda |B| \times \left[1 + \lambda |B| - \left[1 - \exp\{-\lambda |B|\}\right]^{-1} \lambda |B|\right]$$

b)



$$\mathrm{Prob}\{N(\mathcal{B}_i)=j\} = \frac{[\lambda|\mathcal{B}_i|]^j}{j!}\exp\{-\lambda|\mathcal{B}_i|\}$$

$$E\{N(\mathcal{B}_1)\} = \lambda|\mathcal{B}_1|$$

$$E\{N(\mathcal{B}_2)\} = \lambda|\mathcal{B}_2|$$

$$Var\{N(\mathcal{B}_1)\} = \lambda|\mathcal{B}_1|$$

$$Var\{N(\mathcal{B}_2)\} = \lambda|\mathcal{B}_2|$$

$$|\mathcal{B}_i| = A_i \rightarrow N_i$$
$$|\mathcal{B}_i^*| = A_i^* \rightarrow N_i^*$$
$$|\mathcal{B}_{ij}^*| = A_{ij}^* \rightarrow N_{ij}^*$$

$$Cov\{N(\mathcal{B}_1), N(\mathcal{B}_2)\} = Cov\{N_1, N_2\}$$

$$= E\{N_1 N_2\} - E\{N_1\}E\{N_2\}$$

$$= E\{[N_1^* + N_{12}^*][N_2^* + N_{12}^*]\} - E\{N_1\}E\{N_2\}$$

$$E\{N_1^*\}E\{N_2^*\} + E\{N_1^*\}E\{N_{12}^*\} + E\{N_2^*\}E\{N_{12}^*\} + E\{N_{12}^{*2}\}$$
$$+ E\{N_{12}^*\}^2 \qquad - E\{N_{12}^*\}^2$$

$$= [E\{N_1^*\} + E\{N_{12}^*\}][E\{N_2^*\} + E\{N_{12}^*\}] + Var\{N_{12}^*\}$$

$$= E\{N_1\}E\{N_2\} + Var\{N_{12}^*\} - E\{N_1\}E\{N_2\}$$

$$= Var\{N_{12}^*\}$$

$$= Var\{N(\mathcal{B}_{12}^*)\}$$

$$= \lambda|\mathcal{B}_{12}^*| = \underline{\underline{\lambda|\mathcal{B}_1 \cap \mathcal{B}_2|}}$$

For $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$

$$\mathrm{Cov}\{N(\mathcal{B}_1), N(\mathcal{B}_2)\} = \lambda |\mathcal{B}_1 \cap \mathcal{B}_2|$$

$$= \lambda |\emptyset| = 0$$ - uncorrelated

actually independent from def of Poisson RF

For $\mathcal{B}_1 = \mathcal{B}_2$

$$\mathrm{Cov}\{N(\mathcal{B}_1), N(\mathcal{B}_2)\} = \lambda |\mathcal{B}_1 \cap \mathcal{B}_2| = \lambda |\mathcal{B}_1|$$

- same as $\mathrm{Var}\{N(\mathcal{B}_1)\}$ - of course!

c) See Figure on 2b)

$$\text{Prob}\{N(B_2)=i \mid N(B_1)=j\}$$

$$= \text{Prob}\{N_2=i \mid N_1=j\} = \frac{\text{Prob}\{N_2=i,\ N_1=j\}}{\text{Prob}\{N_1=j\}}$$

$$= \frac{\sum_{k=0}^{\min ij} \text{Prob}\{N_{12}^*=k,\ N_2^*=i-k,\ N_1^*=j-k\}}{\text{Prob}\{N_1=j\}}$$

$$= \frac{\sum_{k=0}^{\min ij} \text{Prob}\{N_{12}^*=k\}\,\text{Prob}\{N_2^*=i-k\}\,\text{Prob}\{N_1^*=j-k\}}{\text{Prob}\{N_1=j\}}$$

$$= \left[\frac{[\lambda A_1]^j}{j!}\right]^{-1} \exp\{\lambda A_1\} \times \exp\{-\lambda[A_{12}^* + A_2^* + A_1^*]\}$$
$$\times \sum_{k=0}^{\min ij} \lambda^{k+(i-k)+(j-k)} \frac{A_{12}^{*k} A_2^{*\,i-k} A_1^{*\,j-k}}{k!\,(i-k)!\,(j-k)!}$$

$$= \lambda^{-j} A_1^{-j} j! \times \exp\{-\lambda A_2^*\} \times \lambda^{i+j} \sum_{k=0}^{\min ij} \lambda^{-k} \frac{A_{12}^{*k} A_2^{*\,i-k} A_1^{*\,j-k}}{k!\,(i-k)!\,(j-k)!}$$

$$= \lambda^{i} A_1^{-j} j!\, \exp\{-\lambda A_2^*\} \sum_{k=0}^{\min ij} \lambda^{-k} \frac{A_{12}^{*k} A_2^{*\,i-k} A_1^{*\,j-k}}{k!\,(i-k)!\,(j-k)!}$$

$$= \lambda^{i} |B_1|^{-j} j! \times \exp\{-\lambda |B_2 - [B_1 \cap B_2]|\}$$
$$\times \sum_{k=0}^{\min ij} \lambda^{-k} \frac{|B_1 \cap B_2|^k\, |B_2 - [B_1 \cap B_2]|^{i-k}\, |B_1 - [B_1 \cap B_2]|^{j-k}}{k!\,(i-k)!\,(j-k)!}$$

For $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset \to |\mathcal{B}_1 \cap \mathcal{B}_2| = 0$; $0^k = \begin{cases} 1 & k=0 \\ 0 & k=1,\dots \end{cases}$

$$\text{Prob}\{\mathcal{N}(\mathcal{B}_2) = i \,/\, \mathcal{N}(\mathcal{B}_1) = j\}$$

$$= \lambda^i \, |\mathcal{B}_1|^{-j} \, j! \times \exp\{-\lambda |\mathcal{B}_2|\} \overset{k=0}{\times} \frac{|\mathcal{B}_2|^i \, |\mathcal{B}_1|^j}{i! \, j!}$$

$$= \frac{[\lambda |\mathcal{B}_2|]^i}{i!} \exp\{-\lambda |\mathcal{B}_2|\}$$

$$= \text{Prob}\{\mathcal{N}(\mathcal{B}_2) = i\}$$ — since $[\mathcal{N}(\mathcal{B}_1), \mathcal{N}(\mathcal{B}_2)]$ indep from Prop of Poisson RF.

Problem 3 Mosaic RF

Ising model - Gibbs form:

$$\mathrm{Prob}\{L=l;\beta\} = \mathrm{const} \times \exp\Big\{\beta \sum_{\langle u,v\rangle} I(l_u=l_v)\Big\}$$

Likelihood function

$$\mathrm{Prob}\{O=o \mid L=l;p\} = \prod_{x\in\mathcal{L}_D} \mathrm{Prob}\{O_x=o_x \mid L_x=l_x;p\}$$

misclassification with prob p!

Hence:

$$\mathrm{Prob}\{O_x=o_x \mid L_x=l_x;p\} = [1-p]^{I(o_x=l_x)} \cdot p^{\overbrace{I(o_x\neq l_x)}^{1-I(o_x=l_x)}}$$

and

$$\mathrm{Prob}\{O=o \mid L=l;p\} = p^n \prod_{x\in\mathcal{L}_D}\left[\frac{1-p}{p}\right]^{I(o_x=l_x)}$$

$$= p^n \prod_{x\in\mathcal{L}_D} \exp\Big\{\ln\Big[\frac{1-p}{p}\Big] \times I(o_x=l_x)\Big\}$$

The posterior Gibbs model is:

$$\mathrm{Prob}\{L=l \mid O=o_x\}$$

$$= \mathrm{const} \times p^n \times \exp\Big\{\sum_{x\in\mathcal{L}_D} \ln\Big[\frac{1-p}{p}\Big] I(o_x=l_x) + \beta \sum_{\langle u,v\rangle} I(l_u=l_v)\Big\}$$

$\downarrow$ standard development of Markov model

$$\mathrm{Prob}\{L_x=l_x \mid L_{-x}=l_{-x}, o_x\}$$

$$= \mathrm{const} \times \exp\Big\{\ln\Big[\frac{1-p}{p}\Big] \times I(o_x=l_x) + \beta \sum_{\langle x,v\rangle} I(l_x=l_v)\Big\}$$

$$\forall x\in\mathcal{L}_D$$