Suggested Solution

EXAM - TMA 4250 Spatial Statistics

May 28. 2018 / HENNING OMRE

Problem 1. Continuous RF

$\{r(x); x \in \mathcal{D} \subset \mathbb{R}\}$ - one-dim

$$E\{r(x)\} = \mu_0 + \mu_1 g(x)$$

$$Var\{r(x)\} = \sigma^2 h(x)$$

$$Corr\{r(x'), r(x'')\} = \rho(x''-x') = \rho(\tau) \rightarrow$$

a) Non-negative definite function $c(\tau) \in \mathbb{R}$; $\tau \in \mathbb{R}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j c(x_i - x_j) \geq 0$$

all conf $x_1, \ldots, x_n \in \mathcal{D}$

all weights $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$

all $n \in N$; $n \geq 2$.

If $\rho(\tau)$ is non-negative def & $\rho(0) = 1$

$\Downarrow$

$\rho(\tau)$ valid spatial correlation function

Observations: $\qquad x_0 \qquad \rho(\tau)$

$r(x) \to [r_1, \ldots, r_n] \qquad r_0 \qquad \rho_{0i}$

$g(x) \to [g_1, \ldots, g_n] \qquad g_0 \qquad \rho_{ij}$

$h(x) \to [h_1, \ldots, h_n] \qquad h_0$

b) Known: $\sigma^2, \rho(\tau)$ $\to$ unknown μ_0, μ_1

Predictor:

$$\hat{r}_0 = \sum_{i=1}^{n} \alpha_i r_i$$

Unbiasedness:

$$E\{r_0 - \hat{r}_0\} = 0 \Rightarrow E\{r_0\} - \sum_{i=1}^{n} \alpha_i E\{r_i\} = 0$$

$E\{r_0\} \to \mu_0 + \mu_1 g_0 \qquad E\{r_i\} \to \mu_0 + \mu_1 g_i$

$$= \mu_0\left(1 - \sum_{i=1}^{n} \alpha_i\right) + \mu_1\left(g_0 - \sum_{i=1}^{n} \alpha_i g_i\right) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_i \alpha_i = 1 \qquad\qquad \sum_i \alpha_i g_i = g_0$$

Variance:

$$Var\{r_0 - \hat{r}\} = Var\{r_0 - \sum_i \alpha_i r_i\}$$

$$= Var\{r_0\} - 2\sum_i \alpha_i \, Cov\{r_0, r_i\} + \sum_i \sum_j \alpha_i \alpha_j \, Cov\{r_i, r_j\}$$

$$= \sigma^2 h_0 - 2\sum_i \alpha_i \, \sigma^2 \sqrt{h_0 h_i}\, \rho_{i0} + \sum \sum \alpha_i \alpha_j \sigma^2 \sqrt{h_i h_j}\, \rho_{ij}$$

Minimization:

$$[\hat{\alpha}_1, \ldots, \hat{\alpha}_n] = \underset{[\alpha_1, \ldots, \alpha_n]}{\operatorname{argmin}} \{ Var\{r_0 - \hat{r}\} \}$$

$$\sum_i \alpha_i = 1$$

$$\sum_i \alpha_i g_i = g_0$$

— Lagrange minimization

c) Known: $\sigma^2, \rho(\tau) \rightarrow$ unknown μ_0, μ_1

Estimators:

(A) $\hat{\mu}_0 = \sum_{i=1}^{n} \beta_i^0 r_i$

(B) $\hat{\mu}_1 = \sum_{i=1}^{n} \beta_i^1 r_i$

(A): Unbiasedness:

$$E\{\mu_0 - \hat{\mu}_0\} = 0 \Rightarrow \mu_0 - \sum_i \beta_i^0 \mu_0 - \sum_i \beta_i^0 \mu_1 g_i = 0$$

$$\Rightarrow \mu_0 \left(1 - \sum_i \beta_i^0\right) - \mu_1 \sum_i \beta_i^0 g_i = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_i \beta_i^0 = 1 \qquad \sum \beta_i^0 g_i = 0$$

Variance

$$Var\{\mu_0 - \hat{\mu}_0\} = Var\{\mu_0 - \sum_i \beta_i^0 r_i\}$$

$$= \sum_i \sum_j \beta_i^0 \beta_j^0 \, cov\{r_i, r_j\}$$

$$= \sum_i \sum_j \beta_i^0 \beta_j^0 \sigma^2 \sqrt{h_i h_j} \rho_{ij}$$

Minimization:

$$\left[\hat{\beta}_1^0, \dots, \hat{\beta}_n^0\right] = \underset{[\beta_1^0, \dots, \beta_n^0]}{\operatorname{argmin}} \left\{ Var\{\mu_0 - \hat{\mu}_0\} \right\}$$

$$\sum_i \beta_i^0 = 1 \qquad \text{Lagrange minimization}$$

$$\sum_i \beta_i^0 g_i = 0$$

(B): Unbiasedness:

$$E\{\mu_1 - \hat{\mu}_1\} = 0 \Rightarrow \mu_1 - \sum_i \beta_i^1 \mu_0 - \sum_i \beta_i^1 \mu_1 g_i = 0$$

$$\Rightarrow -\mu_0 \sum_i \beta_i^1 + \mu_1 \left(1 - \sum_i \beta_i^1 g_i\right) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_i \beta_i^1 = 0 \qquad \sum_i \beta_i^1 g_i = 1$$

Variance:

$$Var\{\mu_1 - \hat{\mu}_1\} = Var\{\mu_1 - \sum_i \beta_i^1 r_i\}$$

$$= \sum_i \sum_j \beta_i^1 \beta_j^1 \sigma^2 \sqrt{h_i h_j}\, \rho_{ij}$$

Minimization:

$$\left[\hat{\beta}_1^1, \ldots, \hat{\beta}_n^1\right] = \underset{\substack{\sum_i \beta_i^1 = 0 \\ \sum_i \beta_i^1 g_i = 1}}{\operatorname{argmin}} \{ Var\{\mu_1 - \hat{\mu}_1\}\}$$

Lagrange minimization

Problem 2 Event RF

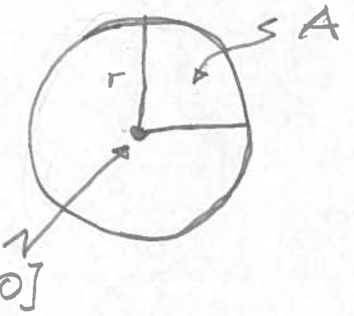
Poisson RF – param intensity $\lambda \geq 0$.

$\{x_i;\ i=1,\ldots,n\}$; $x_i \in \mathcal{D} \subset \mathbb{R}^2$ $n \in \mathbb{N}_\oplus$

$k_{\mathcal{D}}$ – no of points in $\mathcal{D}$

k_A – no of points in A

$\mathcal{D}$:



Known:

a)

$$k_{\mathcal{D}} \Rightarrow p(k_{\mathcal{D}}) = \frac{[\lambda|\mathcal{D}|]^{k_{\mathcal{D}}}}{k_{\mathcal{D}}!} \exp\{-\lambda|\mathcal{D}|\} \quad ; \quad k_{\mathcal{D}} = 0,1,2,\ldots$$

$$E\{k_{\mathcal{D}}\} = \lambda|\mathcal{D}| = \lambda\pi r^2$$

$$Var\{k_{\mathcal{D}}\} = \lambda|\mathcal{D}| = \lambda\pi r^2$$

$$k_A \Rightarrow p(k_A) = \frac{[\lambda|A|]^{k_A}}{k_A!} \exp\{-\lambda|A|\} \quad : \quad k_A = 0,1,2,\ldots$$

$$E\{k_A\} = \lambda|A| = \frac{1}{4}\lambda\pi r^2$$

$$Var\{k_A\} = \lambda|A| = \frac{1}{4}\lambda\pi r^2$$

Note: independent A^c – A complement

$$k_{\mathcal{D}} = k_A + k_{A^c}$$

$$Cov\{k_A, k_{\mathcal{D}}\} = Cov\{k_A, k_A + k_{A^c}\}$$

$$= Cov\{k_A, k_A\} + Cov\{k_A, k_{A^c}\}$$

$$= Var\{k_A\} \qquad = 0$$

$$= \frac{1}{4}\lambda\pi r^2$$

b) (A) $k_{\mathcal{D}}$ unknown — $k_A = k$ known

$$k_{\mathcal{D}} = \overset{k}{\overset{\shortparallel}{k_A}} + k_{A^c}$$

(independent)

$$k_{A^c} \Rightarrow p(k_{A^c}) = \frac{[\lambda |A^c|]^{k_{A^c}}}{k_{A^c}!} \exp\{-\lambda |A^c|\} \; ; \; k_{A^c} = 0, 1, \ldots$$

$\Downarrow$

$$k_{\mathcal{D}} \Rightarrow p(k_{\mathcal{D}}) = \frac{[\lambda |A^c|]^{k_{\mathcal{D}}-k}}{[k_{\mathcal{D}}-k]!} \exp\{-\lambda |A^c|\}$$

$$= \frac{\left[\frac{3}{4}\lambda \pi r^2\right]^{k_{\mathcal{D}}-k}}{(k_{\mathcal{D}}-k)!} \exp\left\{-\frac{3}{4}\lambda \pi r^2\right\} \; ; \; k_{\mathcal{D}} = k, k+1, \ldots$$

(B) k_A unknown — $k_{\mathcal{D}} = k$ known

$$\underset{k}{\underset{\shortparallel}{k_{\mathcal{D}}}} = k_A + k_{A^c}$$

(independent)

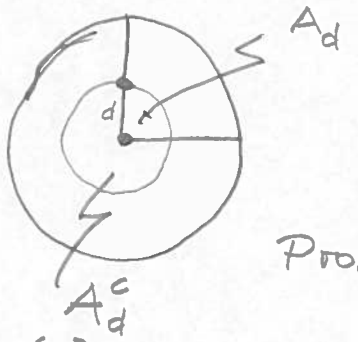
Binomial:

$$k_A \Rightarrow p(k_A) = \begin{bmatrix} k \\ k_A \end{bmatrix} \left[\frac{|A|}{|\mathcal{D}|}\right]^{k_A} \left[\frac{|A^c|}{|\mathcal{D}|}\right]^{k-k_A} \; ; \; k_A = 0, \ldots, k_{\mathcal{D}}$$

$$= \binom{k}{k_A} \left(\frac{1}{4}\right)^{k_A} \left(\frac{3}{4}\right)^{k-k_A}$$

c) k_B unknown − $k_A = k$ known

d_c − dist to closest point from [0,0]



$$\text{Prob}\{d_c > d \mid k_A = k\} \quad k \geq 1$$

Prob{empty A_r} = 0

$$= \text{Prob}\{\text{empty } A_d \cap \text{empty } A_d^c\}$$

$$= \text{Prob}\{\text{empty } A_d\} \times \text{Prob}\{\text{empty } A_d^c\} \qquad 0 < d_c < r$$

$$= \binom{k}{0} \frac{|A_d|}{|A|}^0 \left[1 - \frac{|A_d|}{|A|}\right]^k \times \exp\{-\lambda |A_d^c|\}$$

$$= \left[1 - \frac{\pi d^2}{\pi r^2}\right]^k \times \exp\{-\lambda \tfrac{3}{4}\pi d^2\}$$

$$= \left[1 - \left[\frac{d}{r}\right]^2\right]^k \exp\{-\tfrac{3}{4}\lambda\pi d^2\}$$

$$F(d \mid k_A = k) = \text{Prob}\{d_c < d \mid k_A = k\} = 1 - \left[1 - \left(\frac{d}{r}\right)^2\right]^k \exp\{-\tfrac{3}{4}\lambda\pi d^2\}$$

$$p(d_c \mid k_A = k) = \frac{dF(d \mid k_A = k)}{dd}$$

$$= k\left[1 - \left(\frac{d}{r}\right)^2\right]^{k-1}\left(\frac{-2d}{r^2}\right) \cdot \exp\{-\tfrac{3}{4}\lambda\pi d^2\}$$

$$+ \left[1 - \left(\frac{d}{r}\right)^2\right]^k \cdot \exp\{-\tfrac{3}{4}\lambda\pi d^2\}\left(-\tfrac{3}{2}\lambda\pi d\right)$$

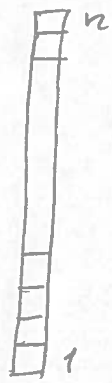
$$= \left[1 - \left(\frac{d}{r}\right)^2\right]^{k-1} \exp\{-\tfrac{3}{4}\lambda\pi d^2\}$$

$$\times \left[\frac{2dk}{r^2} + \left[1 - \left(\frac{d}{r}\right)^2\right]\tfrac{3}{2}\lambda\pi d\right]; \quad 0 < d_c < r$$

$$p(d \mid k_A = k) = 0 \qquad ; \; d \geq r$$

Problem 3. Mosaic RF

Markov random profile



$\mathbf{l} = [l_1, l_2, \ldots, l_n] \quad l_i \in \{w, b\}$

Gibbs formulation:

$$p(\mathbf{l}) = \text{const} \times \prod_{\langle u,v \rangle} \beta^{I(l_u = l_v)}$$

$$= \text{const} \times \prod_{i=1}^{n-1} \beta^{I(l_i = l_{i+1})} \qquad \beta\text{-known}$$

a) Markov formulation:

$$p(l_k \mid \mathbf{l}_{-k}) = \frac{p(\mathbf{l})}{\sum_{l_k'} p(l_k', \mathbf{l}_{-k})} \qquad k = 2, 3, \ldots, n-1$$

$$= \frac{\text{const} \times \prod_{\substack{i=1 \\ i \neq k \\ i \neq k-1}}^{n-1} \beta^{I(l_i = l_{i+1})} \times \beta^{I(l_k = l_{k+1})} \beta^{I(l_{k-1} = l_k)}}{\text{const} \times \prod \cdot \sum_{l_k'} \cdot}$$

$$= \left[\sum_{l_k'} \beta^{I(l_k' = l_{k+1})} \beta^{I(l_{k-1} = l_k')} \right]^{-1} \beta^{I(l_k = l_{k+1})} \beta^{I(l_{k-1} = l_k)}$$

$$= p(l_k \mid l_{k-1}, l_{k+1})$$

$$p(l_1 \mid \mathbf{l}_{-1}) = \frac{p(\mathbf{l})}{\sum_{l_1'} p(l_1', \mathbf{l}_{-1})}$$

$$= \frac{\mathrm{const} \times \prod_{\substack{i=1 \\ i\neq 1}}^{n-1} \beta^{I(l_i = l_{i+1})} \times \beta^{I(l_1 = l_2)}}{\mathrm{const} \prod \cdot \sum_{l_1'} \cdot}$$

$$= \Big[\underbrace{\sum_{l_1'} \beta^{I(l_1' = l_2)}}_{[1+\beta]^{-1}}\Big]^{-1} \beta^{I(l_1 = l_2)}$$

$$= p(l_1 \mid l_2)$$

$$p(l_n \mid \mathbf{l}_{-n}) = \text{similar}$$

$$= \Big[\underbrace{\sum_{l_n'} \beta^{I(l_{n-1} = l_n')}}_{[1+\beta]^{-1}}\Big]^{-1} \beta^{I(l_{n-1} = l_n)}$$

$$= p(l_n \mid l_{n-1})$$

b)

$$p(\mathbf{l}) = p(l_1) \prod_{k=2}^{n} p(l_k | l_{k-1}, \dots, l_1)$$

consider:

$$p(l_k, l_{k-1}, \dots, l_1) = \sum_{l_n'} \cdots \sum_{l_{k+1}'} p(\mathbf{l})$$

$$= \text{const} \prod_{i=1}^{k-1} \beta^{I(l_i = l_{i+1})} \times \underbrace{\sum_{l_n'} \cdots \sum_{l_{k+1}'} \prod_{i=k}^{n-1} \beta^{I(l_i = l_{i+1})}}_{v_{k+1}(l_k)}$$

$$p(l_{k-1}, \dots, l_1) = \sum_{l_k'} p(l_k, l_{k-1}, \dots, l_1)$$

hence

$$p(l_k | l_{k-1}, \dots, l_1) = \frac{p(l_k, l_{k-1}, \dots, l_1)}{p(l_{k-1}, \dots, l_1)}$$

$$= \frac{\text{const} \prod_{i=1}^{k-2} \beta^{I(l_i = l_{i+1})} \beta^{I(l_{k-1} = l_k)} \times v_{k+1}(l_k)}{\text{const} \prod_{i=1}^{k-2} \beta^{I(l_i = l_{i+1})} \underbrace{\sum_{l_k'} \beta^{I(l_{k-1} = l_k')} \times v_{k+1}(l_k)}_{v_k(l_{k-1})}}$$

$$= \left[\sum_{l_k'} \beta^{I(l_{k-1} = l_k')} v_{k+1}(l_k') \right]^{-1} \beta^{I(l_{k-1} = l_k)} v_{k+1}(l_k)$$

$$\equiv \left[v_k(l_{k-1}) \right]^{-1} \beta^{I(l_{k-1} = l_k)} v_{k+1}(l_k) \quad k = 2, \dots, n$$

$$= p(l_k | l_{k-1})$$ – hence a Markov chain!