TMA 4250 SPATIAL STATISTICS

Exam May 22. 2019

Suggested Solution

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Problem 1 Continuous RF

Consider Gauss RF $\{r(x); x \in \mathcal{D} \subset \mathbb{R}^1\}$; $r(x) \in \mathbb{R}$

$E\{r(x)\} = \mu_r$ – unknown

$Var\{r(x)\} = \sigma_r^2$ – known

$Corr\{r(x'), r(x'')\} = \rho_r(\underbrace{\tau}_{x'-x''})$ – known

a) Positive definite function $c(\tau)$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j c(x_i - x_j) \geq 0$$

$\forall\ \boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_n] \in \mathbb{R}^n$

$\forall$ conf $[x_1, \dots, x_n] \in \mathcal{D}^n$

$\forall\ n \in \mathbb{N}_{[1:\infty]}$

Positive definite correlation function $\rho_r(\tau)$:

$\rho_r(0) = 1$

$\rho_r(\tau)$ – positive definite function a $c(\tau)$

Assume: $\rho_r^i(\tau)$; $i = 1, \dots, n_\rho$ pos. def corr func

then

$$\rho(\tau) = \sum_{i=1}^{n} \alpha_i \rho_r^i(\tau) \quad ; \quad \sum_{i=1}^{n} \alpha_i = 1$$

$$\rho(\tau) = \prod_{i=1}^{n} \rho_r^i(\tau)$$

both pos def. correlation functions

Define:

$\{s(x);\ x\in\mathcal{D}\}$ such that

$$[s(x)\,|\,r(x)] = \gamma_{sr}\, r(x) + \epsilon(x)\ ;\ x\in\mathcal{D}$$

$\hookrightarrow$ $E\{\epsilon(x)\}=0$

$Var\{\epsilon(x)\} = \sigma_\epsilon^2$ - known

$Corr\{\epsilon(x'),\epsilon(x'')\} = \rho_\epsilon(\tau)$ - known

$Corr\{\epsilon(x'), r(x'')\} = 0$

b) Parameters of $\{[r(x), s(x)];\ x\in\mathcal{D}\subset\mathbb{R}\}$

$E\{r(x)\} = \mu_r$

$Var\{r(x)\} = \sigma_r^2$

$Cov\{r(x'), r(x'')\} = \sigma_r^2\rho_r(\tau)$

$E\{s(x)\} = \gamma_{sr}\mu_r$

$Var\{s(x)\} = \gamma_{sr}^2\sigma_r^2 + \sigma_\epsilon^2$

$Cov\{s(x'), s(x'')\}$
$= \gamma_{sr}^2\sigma_r^2\rho_r(\tau) + \sigma_\epsilon^2\rho_\epsilon(\tau)$

$Cov\{r(x), s(x)\} = \gamma_{sr}\sigma_r^2$

$Cov\{r(x'), s(x'')\} = \gamma_{sr}\sigma_r^2\rho_r(\tau)$

Observations - exact

$[r(x_1^{or}), r(x_2^{or})]$ $\quad$ $[s(x_1^{os}), s(x_2^{os})]$

different locations

c) Consider arbitrary location $x_0 \in \mathcal{D}$, and linear predictor

$$\hat{r}(x_0) = \sum_{i=1}^{2} \alpha_i r(x_i^{or}) + \sum_{i=1}^{2} \beta_i s(x_i^{os})$$

Define BLUP system:

Unbiasedness:

$$E\{r(x_0) - \hat{r}(x_0)\} = 0$$

$$\mu_r = \sum_{i=1}^{2} \alpha_i \mu_r + \sum_{i=1}^{2} \beta_i \gamma_{sr} \mu_r$$

$$\longrightarrow \sum_{i=1}^{2} \alpha_i + \gamma_{sr} \sum_{i=1}^{2} \beta_i = 1$$

Prediction variance

$$Var\{r(x_0) - \hat{r}(x_0)\} = Var\{r(x_0) - \sum_i \alpha_i r(x_i^{or}) - \sum_i \beta_i s(x_i^{os})\}$$

$$= \sigma_r^2 + \sum_i \sum_j \alpha_i \alpha_j \sigma_r^2 \rho(\tau_{ij}^r) + \sum_i \sum_j \beta_i \beta_j [\gamma_{sr}^2 \sigma_r^2 \rho_r(\tau_{ij}^s) + \sigma_\epsilon^2 \rho_\epsilon(\tau_{ij}^s)]$$

$$- 2 \sum_i \alpha_i \sigma_r^2 \rho_r(\tau_{0i}^r) - 2 \sum_i \beta_i \gamma_{sr} \sigma_r^2 \rho_r(\tau_{0i}^s)$$

$$+ \sum_i \sum_j \alpha_i \beta_j \gamma_{sr} \sigma_r^2 \rho_r(\tau_{ij}^{sr})$$

BLUE-system:

$$[\hat{\alpha}, \hat{\beta}] = \underset{\alpha, \beta}{\operatorname{argmin}} \{Var\{r(x_0) - \hat{r}(x_0)\}\}$$

$$\sum_i \alpha_i + \gamma_{sr} \sum_i \beta_i = 1$$

Problem 2 Event RF

consider $\mathbb{P}$ of area $a_p = \pi r^2$

$$\mathbb{X}_{\mathbb{P}} : \{ x_i ; i = 1, \ldots, k_p \} \quad x_i \in \mathcal{P} \subset \mathbb{R}^2, \; k_p \in \mathbb{N}_{\oplus}$$

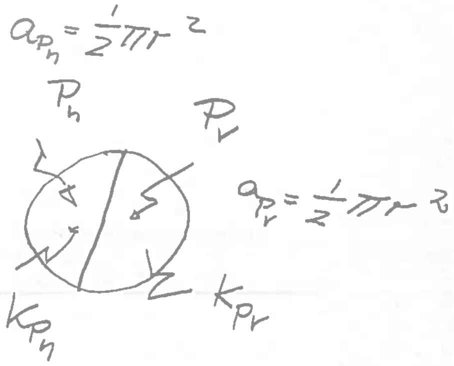
and being a stationary Poisson random field.

a) Expression:

$$p(x_1, \ldots, x_{k_p}) = p(x_1, \ldots, x_k \mid k_p = k)\, p(k_p = k)$$

$$= \prod_{i=1}^{k} \frac{1}{a_p} \times \frac{[\lambda_0 a_p]^k}{k!} \exp\{-\lambda_0 a_p\} \qquad \begin{array}{l} x_i \in \mathcal{P} \\ k_p \in \mathbb{N}_{\oplus} \end{array}$$

$$= \frac{\lambda_0^k}{k!} \exp\{-\lambda_0 a_p\}$$



b)

$$E\{k_p\} = \lambda_0 a_p$$

$$E\{k_p \mid k_{pr} = k_h\}$$

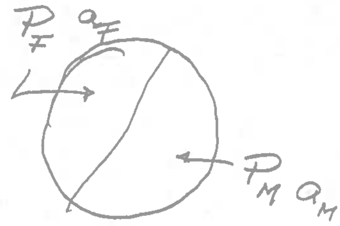
$$= E\{k_{P_h}\} + k_h = \lambda_0 \tfrac{1}{2} \pi r^2 + k_h$$

c) Experience:

$$p(\nu_M, \nu_F; \alpha, \beta) = \text{const}_{**} \times \nu_M^{\alpha-1} \nu_F^{\beta-1}$$

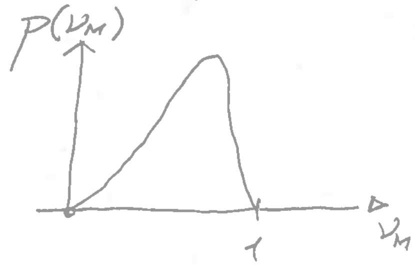
$$E\{a_M\} = a_p E\{\nu_M\} = a_p \frac{\alpha}{\alpha+\beta}$$

$$E\{a_F\} = a_p E\{\nu_F\} = a_p \frac{\beta}{\alpha+\beta}$$



$\nu_F = a_F / a_p$

$\nu_M = a_M / a_p$ $\qquad \nu_F + \nu_M = 1$



Beta pdf!

Observations k_M and k_F, use Bayes rule:

$$p(\nu_M, \nu_F / k_M, k_F)$$

$$= \text{const} \times p(k_M, k_F / \nu_M, \nu_F) \times p(\nu_M, \nu_F)$$

Note:

$$p(k_M, k_F / \nu_M, \nu_F) = p(k_M / \nu_M)\, p(k_F / \nu_F) \qquad a_F \cap a_M = \phi$$

$$= \frac{(\lambda_0 a_p \nu_M)^{k_M}}{k_M!} \exp\{-\lambda_0 a_p \nu_M\} \times \frac{(\lambda_0 a_p \nu_F)^{k_F}}{k_F!} \exp\{-\lambda_0 a_p \nu_F\}$$

$$= \underbrace{\frac{(\lambda_0 a_p)^{k_M}}{k_M!} \times \frac{(\lambda_0 a_p)^{k_F}}{k_F!} \exp\{-\lambda_0 a_p \underbrace{(\nu_M + \nu_F)}_{=1}\}}_{\substack{\text{indep of } \nu_M, \nu_F \\ \text{const}_*}} \times \nu_M^{k_M} \nu_F^{k_F}$$

then

$$p(\nu_M, \nu_F / k_M, k_F) = \text{const} \times \text{const}_* \, \nu_M^{k_M} \nu_F^{k_F} \times \text{const}_{**} \times \nu_M^{\alpha-1} \nu_F^{\beta-1}$$

$$= \text{const}_+ \times \nu_M^{\alpha+k_M-1} \nu_F^{\beta+k_F-1}$$

hence

$$E\{\nu_M / k_M, k_F\} = \frac{\alpha + k_M}{\alpha+\beta+k_M+k_F}$$

$$E\{\nu_F / k_M, k_F\} = \frac{\beta + k_F}{\alpha+\beta+k_M+k_F}$$

$$E\{a_M / k_M, k_F\} = a_p \cdot \frac{\alpha + k_M}{\alpha+\beta+k_M+k_F}$$

$$E\{a_F / k_M, k_F\} = a_p \frac{\beta + k_F}{\alpha+\beta+k_M+k_F}$$

Problem 3 Mosaic Random Field

Markov RF $\mathbf{l}: \{l_x ; x \in \mathcal{L}_D\}$; $l_x \in \Omega_l : \{B, W\}$

$\uparrow$ lattice over $D \subset \mathbb{R}^2$

Gibbs formulation:

$$p(\mathbf{l}) = \text{const} \times \prod_{\langle u,v \rangle} \beta^{I(l_u = l_v)}$$

$\uparrow$ known $\beta \geq 1$

Observations $\mathbf{d}: \{d_x ; x \in \mathcal{L}_D\}$; $d_x \in \mathbb{R}$

Likelihood model:

$$p(\mathbf{d} | \mathbf{l})$$

Posterior model:

$$p(\mathbf{l} | \mathbf{d}) = \text{const} \times p(\mathbf{d} | \mathbf{l}) \, p(\mathbf{l})$$

a) Consider likelihood model:

$$p(\mathbf{d} | \mathbf{l}) = \prod_{x \in \mathcal{L}_D} p(d_x | \mathbf{l}) = \prod_{x \in \mathcal{L}_D} p(d_x | l_x)$$

Posterior model:

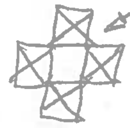
Gibbs form:

$$p(\mathbf{l} | \mathbf{d}) = \text{const} \times \prod_{x \in \mathcal{L}_D} p(d_x | l_x) \prod_{\langle u,v \rangle} \beta^{I(l_u = l_v)}$$

Markov form:

$$p(l_x \mid \mathbf{l}_{-x}, \mathbf{d}) = \frac{p([l_x, \mathbf{l}_{-x}] \mid \mathbf{d})}{\sum\limits_{l_x' \in \Omega_l} p(l_x', \mathbf{l}_{-x} \mid \mathbf{d})}$$

$$= \frac{\text{const} \times \prod\limits_{\substack{y \in \mathcal{L}_D \\ y \neq x}} p(d_y \mid l_y) \prod\limits_{\substack{\langle u,v \rangle \\ u \neq x \\ v \neq x}} \beta^{I(l_u = l_v)} \times p(d_x \mid l_x) \prod\limits_{y \in n_x} \beta^{I(l_x = l_y)}}{\text{const} \times \prod\limits_{\substack{y \in \mathcal{L}_D \\ y \neq x}} \cdot \prod\limits_{\substack{\langle u,v \rangle \\ u \neq x \\ v \neq x}} \cdot \sum\limits_{l_x' \in \Omega_l} p(d_x \mid l_x') \prod\limits_{y \in n_x} \beta^{I(l_x' = l_y)}}$$

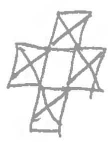


neighbor-hood of x

$n_x = \{ z \in \mathcal{L}_D \mid \langle x, z \rangle \}$

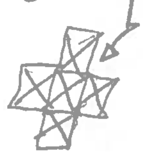
$$= \left[\sum_{l_x' \in \Omega_l} p(d_x \mid l_x') \prod_{y \in n_x} \beta^{I(l_x' = l_y)} \right]^{-1} p(d_x \mid l_x) \prod_{y \in n_x} \beta^{I(l_x = l_y)}$$

$$= p(l_x \mid l_y; y \in n_x, d_x)$$



b) Consider likelihood model

$$p(d\!\!l \mid l\!\!l) = \prod_{x\in\mathcal{L}_D} p(d_x \mid l\!\!l) = \prod_{x\in\mathcal{L}_D} p(d_x \mid l_y;\, y\in\partial_n)$$



Posterior model:

Gibbs form:

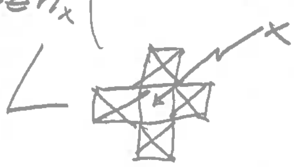
$$p(l\!\!l \mid d\!\!l) = \text{const} \times \prod_{x\in\mathcal{L}_D} p(d_x \mid l_y;\, y\in\partial_n) \prod_{\langle u,v\rangle} \beta^{I(l_u=l_v)}$$

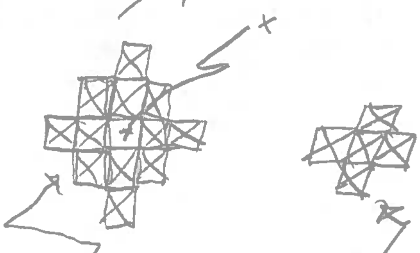
Markov form:

$$p(l_x \mid l\!\!l_{-x};\, d\!\!l) = \frac{p([l_x, l\!\!l_{-x}] \mid d\!\!l)}{\sum_{l'_x\in\Omega_l} p([l'_x, l\!\!l_{-x}] \mid d\!\!l)}$$

$$= \frac{\text{const} \times \prod_{\substack{y\in\mathcal{L}_D\\ x\notin\partial_y}} p(d_y \mid l_z;\, z\in\partial_y) \prod_{\substack{\langle u,v\rangle\\ u\neq x\\ v\neq x}} \beta^{I(l_u=l_v)} \times \prod_{x\in\partial_y} p(d_y \mid l_z;\, z\in\partial_y) \prod_{y\in n_x} \beta^{I(l_x=l_y)}}{\text{const} \times \prod_{\substack{y\in\mathcal{L}_D\\ x\in\partial_y}} \cdot \prod_{\substack{\langle u,v\rangle\\ u\neq x\\ v\neq x}} \cdot \sum_{l'_x\in\Omega_l} \prod_{x\in\partial_y} p(d_y \mid l'_x, l_z;\, z\in\partial_y \setminus x) \prod_{y\in n_x} \beta^{I(l'_x=l_y)}}$$

$$= \Big[\sum_{l'_x\in\Omega_l} \prod_{x\in\partial_y} \cdot \prod_{y\in n_x} \cdot \Big]^{-1} \times \prod_{x\in\partial_y} p(d_y \mid l_z;\, z\in\partial_y) \prod_{y\in n_x} \beta^{I(l'_x=l_y)}$$





$$= p(l_x \mid l_y;\, y\in w_x;\, d_y;\, y\in m_x)$$