Spatial Statistics
Assignment

Henning Omre

September 2004

1 Part I: Continous Random Fields

The R/Spatial Library is required for solving the problems. The command `library(spatial)` loads the library.

Problem 1: 1D Gaussian RF

Consider a Gaussian RF $\{R(x); x \in \mathcal{D} \subset \mathcal{R}^1\}; \mathcal{D} \in [0, 50]$ with

$$
\begin{aligned}
E\{R(x)\} &= 0 \\
Cov\{R(x'), R(x'')\} &= \sigma^2 \times \exp\{-\frac{|x' - x''|^\nu}{\beta}\}
\end{aligned}
\tag{1}
$$

where σ^2, β and ν are model parameters.

(a) Plot the spatial covariance function for different values of the parameters, start with $(\sigma^2, \beta, \nu) = (1, 3, 1)$. Describe the influence of each of the parameters on the Gaussian RF.

The spatial variogram function is defined as

$$
\gamma(x' - x'') = \frac{1}{2} Var\{R(x') - R(x'')\} \tag{2}
$$

Develop the expression for the variogram function for the RF specified above, and plot it for one choice of parameter values. Explain which features are denoted sill and range.

Hint: Use the functions `expcov` and `gaucov`.

(b) Discretize $\mathcal{D}$ into $\mathcal{L}_{\mathcal{D}} : [0, 1, \dots, 50]$ and define the discretized Gaussian RF $\{R(x); x \in \mathcal{L}_{\mathcal{D}}\}$. Use the definition of Gaussian RF to simulate this discretized RF for different values of the model parameters. Comment on the results.

Hint: Use the function `mvrnorm` in the `MASS` library.

(c) Use the model parameters $(\sigma^2, \beta, \nu) = (4, 3, 1)$ and simulate one realization of $\{R(x); x \in \mathcal{L}_{\mathcal{D}}\}$. Plot the realization and compute the area under the realization and above level 2.0.

Let the realizations in locations $x = (10, 15, 40)$ be considered as observations o.

Consider the discretized conditional Gaussian RF $\{[R(x)|o]; x \in \mathcal{L}_{\mathcal{D}}\}$ and generate 100 realizations of it. Plot 10 of these realizations, and comment.

In each location $x \in [0, 1, \dots, 50]$ compute the average and empirical variance over the 100 realizations. Plot the results, and comment.

Compute the discretized expected function $\{E\{R(x)|o\}; x \in \mathcal{L}_{\mathcal{D}}\}$ and variance function $\{Var\{R(x)|o\}; x \in \mathcal{L}_{\mathcal{D}}\}$, plot the results, and comment.

Compare the results obtained above.

Problem 2: Kriging Prediction

Consider the terrain elevation in an area $\mathcal{D} : [(0, 0) \times (7, 7)]$, denoted $\{r(x); x \in \mathcal{D} \subset \mathcal{R}^2\}$. The terrain elevation is observed in 52 locations, $r^d = (r(x_1), r(x_2), \dots, r(x_{52}))$, which are available in the `MASS` library. Type `data(topo)` to obtain the data

(a) Let the terrain elevations be modelled as an isotropic continuous RF $\{R(x); x \in \mathcal{D}\}$ with

$$
\begin{aligned}
R(x) &= \sum_{r+s \leq p} a_{rs}(x^1)^r (x^2)^s + U(x) \\
Cov\{U(x'), U(x'')\} &= exp\{-0.7|x' - x''|\}
\end{aligned}
\quad (3)
$$

where $x = (x^1, x^2)$ and a_{rs} are unknown coefficients.

Based on the model assumptions and the available observations, r^d , compute the Universal Kriging (UK) prediction for a suitable discrtization over $\mathcal{D}$. Use $p = 0, 2, 4, 6$ in the calculations. Compute also the associated UK prediction standard deviation.

Plot the results as contour plots, gray-scale plots and perspective plots with the observations marked. Comment on the results.

Hint: Use the function `surf.gls` and `semat` for UK and the functions `contour`, `image` and `persp` for plotting.

(b) Alternatively one may use an ordinary regression approach, termed trend surface, with

$$\begin{aligned} R(x) &= \sum_{r+s \leq p} a_{rs}(x^1)^r(x^2)^s + U(x) \\ Cov\{U(x'), U(x'')\} &= \mathrm{I}(x' = x'') \end{aligned} \tag{4}$$

where $x = (x^1, x^2)$, $\mathrm{I}(A)$ is equal to 1 whenever A is true an d zero else, and a_{rs} are unknown coefficients. Note that the residuals do not have any spatial dependence.

Based on these model assumptions and the available observations, r^d, compute the trend surface for a suitable discretization over $\mathcal{D}$. Use $p = 0, 2, 4, 6$ in the calculations.

Plot the results as in point a) and comment.

Hint: Use the function `surf.ls` for trend surface analysis.

(c) Discuss the following:

- similarities and differences for UK and trend surface
- strengths and weaknesses for each approach
- strengths and weaknesses of various display alternatives

Part II: Event RF

Problem 3: Gibbs RF/Strauss model

Consider an Event RF $\{X_i; i = 1, \ldots, n; \mathcal{D}\}$ given number of points equal n. Let it be represented by a Gibbs pdf:

$$f(x|n) = const \times \exp\{-\sum_i^n \sum_j^n \phi(\tau_{ij})\} \tag{5}$$

where $x = (x_1, \ldots, x_n), \tau_{ij} = |x_i - x_j|$ and $\phi(\tau)$ is the interaction function,

$$\phi(\tau) = \begin{cases} \alpha & \tau < r \\ 0 & \text{else} \end{cases} \tag{6}$$

This model is termed the Strauss model.

(a) Use the MCMC-MH algorithm to generate a realization of this Strauss model on a unit square with $n = 50, r = 0.1$ and $\alpha = 10$. Repeat the simulations for $\alpha = 1$ and $\alpha = 0$. Justify the convergence of the algorithm.

Plot the results and comment.

(b) Estimate the L-function for each of the realizations in point a).

Simulate 100 realizations of a Binominal RF on the unit square with $n = 50$. Estimate the expected and 0.8 envelope of the L-function for this Binominal RF based on these realizations.

Plot the L-functions estimated above and comment on the results.

(c) Assume that the number of points is random according to a Poisson distribution with parameter λ. Specify the expression for $f(x, n)$.

Specify a simulation algorithm for the corresponding Event RF - the algorithm need not be implemented.

Part III: Markov Fields

Problem 5: Markov RF

Consider a binary Markov RF $L : \{L_x; x \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a lattice over $\mathcal{D} \subset \mathcal{R}^2$ and $L_x \in [-1, 1]$. The pdf is:

$$Prob\{L = l\} = const \times \exp\{\sum_{x \sim y}[\beta_- I(l_x = l_y = -1) + \beta_+ I(l_x = l_y = 1)]\} \tag{7}$$

where $x \sim y$ indicates sum over closest neighbours, $I(A)$ is one whenever A is true and zero otherwise, and (β_-, β_+) are model parameters. Note that by setting $\beta_- = \beta_+$ one obtains the Ising model.

(a) Specify the neighborhood and clique systems of the model.

Simulate three realizations on lattice $[32 \times 32]$ by the MCMC-MH algorithm with model parameters $\beta_- = \beta_+ = 0.23$. Repeat the simulations for three other choices of model parameters. Justify the convergence of the algorithm.

Plot the results and comment.

Hint: Use Matlab in the implementation of the simulation algorithm.

(b) Specify the pseudolikelihood function for the parameters in the model.

Use maximum pseudolikelihood to estimate (β_-, β_+) from each of the realizations generated in point a). Comment on the results.

Problem 6: Bayesian image analysis

The noisy image $o : \{o_x; x \in \mathcal{L}_\mathcal{D}\}$ with $\mathcal{L}_\mathcal{D} : [115 \times 52]$ and $o_x \in \mathcal{R}^1$ can be copied from the file noiseimage.mat. Assume that the likelihood model is:

$$O : \{O_x = l_x + U_x; x \in \mathcal{L}_\mathcal{D}\} \tag{8}$$

where $l_x \in \{-1, 1\}$ and $U : \{U_x; x \in \mathcal{L}_\mathcal{D}\}$ are iid $Gauss_1(u_x; 0, 1.5^2)$.

Assume further that a representative prior model for $l : \{l_x; x \in \mathcal{L}_\mathcal{D}\}$ is a Markov RF as

$$Prob\{L = l\} = const \times \exp\{-\frac{1}{2}\sum_{x \sim y} I(l_x = l_y)\} \tag{9}$$

where $x \sim y$ indicates summation over closest neighbours and $I(A)$ equals one whenever A is true and zero else.

(a) Specify the posterior model for $[L|o]$.

Demonstrate that the posterior model is also a Markov RF with the same neighborhood system as the prior model.

(b) Use the MCMC-Gibbs algorithm to generate realizations from the posterior model. Use torus border conditions in the algorithm. Justify that the algorithm has converged.

Plot some of the realizations and comment.

(c) Define the Locationwise maximum posterior (LMAP) predictor of the image.

Estimate LMAP for the image above, plot it and comment.