SIF 5064 Spatial Statistics,
Assignment 3: Markov fields

Exercise 1: Simulation and estimation of parameters

We study a first order Markov random field (on a regular grid). The joint probabilty distribution function is given by:

$$f(x) = C.exp\{\sum_{i\sim j} (\beta_0 I(x_i = 0, x_j = 0) + \beta_1 I(x_i = 1, x_j = 1))\}$$

where $x = (x_1, \ldots, x_n)$ and $x_i \in \{0, 1\}$. Note that this is a generalization of the Ising model, which occurs by setting $\beta_0 = \beta_1$.

a. Specify and implement an McMC algorithm for this model. [Note: Splus has problems with loops. You may use matlab]. Generate realizations from three choices of parameter value, one equal to $\beta_0 = \beta_1 = 0.23$.

[Note: It takes at least a few minutes to genrate realizations on, say, a 32×32 grid in Matlab. So, be careful with your codes. Matlab contains a routine called 'profile' which gives information on time spent (and where) in the your program. Type 'help profile' to read about it, or ask for help.]

b. Evaluate the pseudo-likelihood for the model. Use the realization obatined with $\beta_0 = \beta_1 = 0.23$, and estimate β_0 and β_1 using pseudo-likelihood. [Use Splus or Matlab. In Matlab, there is a function called 'fmins' well suited for this.]

Exercise 2: Bayesian image analysis

Suppose measurements (observations) of a given picture, $y = \{y_{ij}; i, j = 1, 2, \ldots, n\}$, with $y_{ij} \in (-\infty, \infty)$, are available. We want to reconstruct the true, underlying image, $x = \{x_{ij}; i, j = 1, 2, \ldots, n\}$. In doing so, we formulate a Bayesian model:

Assume x is a binary image; $x_{ij} \in \{0, 1\}$, generated from an Ising model with fixed and known parameter β. Assume further that y is given from x by smoothing operation, with noise:

$$y_{ij} = \frac{1}{q} \sum_{(k,l) \in A_{ij}} x_{kl} + \epsilon_{kl}$$

where $A_{ij} = \{(k,l) :\mid k-i \mid \leq 1, \mid l-j \mid \leq 1\}$. In other words, A_{ij} consists of 3×3 pixels centered at (i,j). (If (i,j) is on the edge of the image, A_{ij} consists of less pixels.) $\epsilon_{kl} \sim N(0, \sigma^2)$, with σ^2 fixed and known, and the ϵ_{kl} are independent.

a. Evaluate (write down) the prior distribution, $f(x)$, and the likelihood, $f(y \mid x)$, from the above assumptions.

b. Show that the posterior, $f(x \mid y)$, is also a Markov filed. Specify the neighborhood in the posterior, and the potential function. (Do this only for pixels away from the edges.)

c. Let $\hat{x} = E[x \mid y]$ be your preferred estimator of the undelying image. Describe how you would find this computationally. (You do not have to implement it at this point.)