Solutions. TMA4255, 2018 (August)

1.

a)

$$F = \frac{MSR}{MSE} = \frac{SSR/DF}{SSE/DF} = \frac{20.786}{4.404} = 4.72.$$

This is a test statistic of for the hypothesis

$H_0 : \beta_1 = \beta_2 = 0$

H_1 : not both are 0.

Since the displayed $P-$value os 0.05, H_0 is rejected if the significance level is greater than or equal 0.05 and is not rejected if the significance level is less than 0.05.

b) The test statistic is

$$T_2 = \frac{\hat{\beta}_2}{\mathrm{SE}(\hat{\beta}_2)}.$$

Under H_0,

$$T_2 \sim t_{0.05,10-2-1} = t_{0.05,7}.$$

H_0 is rejected if $T_2 < -t_{0.05,7}$. We have $T_2 = -2.88$, $t_{0.05,7} = -1.895$, so H_0 is rejected.

Since this is a one-sided test, the $P-$value is $0.024/2 = 0.012$.

c) $S = 2.09859$. First, let us find the confidence interval for σ^2. Use that

$$\frac{SSE}{\sigma^2} \sim \chi^2_7$$

therefore

$$P\left(1.690 \le \frac{SSE}{\sigma^2} \le 16.013\right) = 0.95$$

or

$$P\left(\frac{SSE}{16.013} \le \sigma^2 \le \frac{SSE}{1.669}\right) = 0.95,$$

and the confidence interval for σ^2 is

$$[1.9252, 18.242].$$

Thus the confidence interval for σ is

$$[1.3875, 4.2711].$$

The hypothesis $H_0 : \sigma = 1$ is tested versus the alternative $H_1 : \sigma \neq 1$ with significance level 0.05 (i.e. $1 - 0.95$). H_0 is rejected if the confidence interval does not contain 1.

d) The derivative of $10.1 + 7.36x - 1.14x^2$ is $7.36 - 2.28x$. It equals 0 when

$$x = \frac{7.36}{2.28} = 3.23.$$

Point prediction:

$$\hat{y}_0 = 10.1 + 7.36 \cdot 3.23 - 1.14 \cdot 3.23^2 = 21.98.$$

$$\mathrm{SD}(\hat{y}_0) = S\sqrt{\mathbf{x}_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_0},$$

therefore

$$\mathbf{x}_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_0 = \left(\frac{1.024}{2.09859}\right)^2 = 0.238.$$

The prediction interval is

$$21.98 \pm 2.365 \cdot 2.09859\sqrt{1+0.238}$$

i.e. $[16.46, 27.5]$.

2.

a) Let X be the number of hits. X has the binomial distribution with parameters (n, p) where $n = 20$. The following hypothesis is tested

$$H_0 : p = p_0 = 0.6, \ H_1 : p > p_0 = 0.6.$$

Under H_0 the test statistic

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}}$$

has the standard normal distribution (approximately). H_0 is rejected if $Z \geq z_\alpha$. In our case $z_\alpha = 1.645$ and the observed value of Z is 2.7. H_0 is rejected.

b) The $p-$value is equal to $P(X \geq 18)$ where X has the binomial distribution with parameters (20,0.6). This probability can be calculated either directly (exactly) or using the normal approximation.

1. Directly

$$P(X \geq 18) = \frac{20!}{18!2!}0.6^{18}0.4^2 + \frac{20!}{19!1!}0.6^{19}0.4^1 + \frac{20!}{20!0!}0.6^{20}0.4^0 = 0.0035.$$

2. Using normal approximation

$$P(X \geq 18) = P\left(Z \geq \frac{18 - 20 \cdot 0.6}{\sqrt{20 \cdot 0.6 \cdot 0.4}}\right) =$$

$$= P(Z \geq 2.7) = P(Z \leq -2.7) = 0.0035.$$

3.

a) The hypothesis

$$H_0 : \mu_1 = \mu_2$$

is tested versus the alternative

$$H_1 : \mu_1 > \mu_2.$$

We can use the pared $t-$test. The test statistic

$$T = \sqrt{n}\frac{\bar{D}}{S_D}.$$

The observed value of the test statistic is

$$t_{\text{obs}} = \sqrt{15}\frac{21.53}{38.29} = 2.177.$$

This is a one-sided test, the null hypothesis is rejected if $t_{\text{obs}} \geq t_{\alpha,n-1}$. In our case

$$t_{\text{obs}} = 2.177 > 1.761 = t_{0.05,14}.$$

The null hypothesis is rejected.

b) We use the sign test. Let D be the number of differences $D_i = X_{1i} - X_{2i}$ which are positive. The null hypothesis H_0 is rejected if the test statistic

$$\frac{D - n/2}{\sqrt{n/4}},$$

which under H_0 has (approximately) the standard normal distribution, is greater than z_α. In our case, $z_\alpha = 1.645$, and the observed value of the test statistic is

$$\frac{13 - 7.5}{\sqrt{15/4}} = 2.84.$$

H_0 is rejected.