Chapter 13

• Multiple comparisons. Tukey's test. $n_1 = n_2 = ... = n_k = n$. Testing simultaneously $H_0 : \mu_i = \mu_j$ for all $i \neq j$. Denote

$$I_{ij} = \left[\bar{Y}_{i.} - \bar{Y}_{j.} - q(\alpha, k, N-k)\frac{S}{\sqrt{n}}, \bar{Y}_{i.} - \bar{Y}_{j.} + q(\alpha, k, N-k)\frac{S}{\sqrt{n}}\right].$$

If $0 \notin I_{ij}$, then $H_0 : \mu_i = \mu_j$ is rejected.

• Randomized complete block design. Model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij},\ i = 1, 2, ..., k,\ j = 1, 2, ..., b.$$

α_i is the effect of the i-th treatment, β_j is the effect of the j-th block, errors ϵ_{ij} are independent, $\epsilon_{ij} \sim N(0, \sigma^2)$. Hypothesis

$$H_0 : \alpha_1 = \alpha_2 = ... = \alpha_k = 0$$

$$H_1 : \alpha_i \neq 0 \text{ for at least one } i$$

• Means and sums of squares:

$$\bar{Y}_{i.} = \frac{1}{b}\sum_{j=1}^{b} Y_{ij},\ \bar{Y}_{.j} = \frac{1}{k}\sum_{i=1}^{k} Y_{ij},\ \bar{Y}_{..} = \frac{1}{kb}\sum_{i=1}^{k}\sum_{j=1}^{b} Y_{ij},$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{..})^2$$

(total sum of squares),

$$SSA = \sum_{i=1}^{k}\sum_{j=1}^{b}(\bar{Y}_{i.} - \bar{Y}_{..})^2 = b\sum_{i=1}^{k}(\bar{Y}_{i.} - \bar{Y}_{..})^2$$

(treatment sum of squares),

$$SSB = \sum_{i=1}^{k}\sum_{j=1}^{b}(\bar{Y}_{.j} - \bar{Y}_{..})^2 = k\sum_{j=1}^{b}(\bar{Y}_{.j} - \bar{Y}_{..})^2$$

(block sum of squares),

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2$$

(error sum of squares).

• Under H_0

$$F = \frac{SSA/(k-1)}{SSE/(k-1)(b-1)} \sim F_{k-1,(k-1)(b-1)}.$$

Test: if $F \geq f_{\alpha,k-1,(k-1)(b-1)}$, then H_0 is rejected.

Chapter 14

• Two-factor ANOVA. Two factors A and B. A has a levels, B has b levels. There can be interactions. Model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk},\ i = 1, 2, ..., a,\ j = 1, 2, ..., b,\ k = 1, 2, ..., n,$$

where ϵ-s are independent and $\sim N(0, \sigma^2)$, and

$$\sum_{i=1}^{a} \alpha_i = 0,\ \sum_{j=1}^{b} \beta_j = 0,\ \sum_{i=1}^{a} (\alpha\beta)_{ij} = 0,\ \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0.$$

• The three hypotheses:

1.
$$H_0' : \alpha_1 = \alpha_2 = ... = \alpha_a = 0$$
$$H_1' : \alpha_i \neq 0 \text{ for at least one } i$$

2.
$$H_0'' : \beta_1 = \beta_2 = ... = \beta_b = 0$$
$$H_1'' : \beta_j \neq 0 \text{ for at least one } j$$

3.
$$H_0''' : (\alpha\beta)_{11} = (\alpha\beta)_{12} = ... = (\alpha\beta)_{ab} = 0$$
$$H_1''' : (\alpha\beta)_{ij} \neq 0 \text{ for at least one } (i, j)$$

• Means:

$$\bar{Y}_{ij.} = \frac{1}{n} \sum_{k=1}^{n} Y_{ijk},\ \bar{Y}_{i..} = \frac{1}{nb} \sum_{j=1}^{b} \sum_{k=1}^{n} Y_{ijk},\ \bar{Y}_{.j.} = \frac{1}{na} \sum_{i=1}^{a} \sum_{k=1}^{n} Y_{ijk},$$

$$\bar{Y}_{...} = \frac{1}{nab} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} Y_{ijk}.$$

Sums of squares:

$$SSA = bn \sum_{i=1}^{a} (\bar{Y}_{i..} - \bar{Y}_{...})^2$$

(sum of squares for factor A),

$$SSB = an \sum_{j=1}^{b} (\bar{Y}_{.j.} - \bar{Y}_{...})^2$$

(sum of squares for factor B),

$$SS(AB) = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$$

(interaction sum of squares),

$$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk} - \bar{Y}_{ij.})^2$$

(error sum of squares),

$$SST = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk} - \bar{Y}_{...})^2$$

(total sum of squares).

- Sum-of-squares identity

$$SST = SSA + SSB + SS(AB) + SSE.$$

- $SSA,\ SSB,\ SS(AB),\ SSE$ are independent and

$$\frac{SSE}{\sigma^2} \sim \chi^2_{ab(n-1)},$$

$$\frac{SSA}{\sigma^2} \sim \chi^2_{a-1} \text{ under } H_0',$$

$$\frac{SSB}{\sigma^2} \sim \chi^2_{b-1} \text{ under } H_0'',$$

$$\frac{SS(AB)}{\sigma^2} \sim \chi^2_{(a-1)(b-1)} \text{ under } H_0''',$$

- 1. Test statistic

$$F_1 = \frac{SSA/(a-1)}{SSE/ab(n-1)}$$

has (under H_0') F-distribution with $a-1$ and $ab(n-1)$ degrees of freedom. If $F_1 > f_{\alpha,a-1,ab(n-1)}$, then H_0' is rejected.

2. Test statistic

$$F_2 = \frac{SSB/(b-1)}{SSE/ab(n-1)}$$

has (under H_0'') F-distribution with $b-1$ and $ab(n-1)$ degrees of freedom. If $F_2 > f_{\alpha,b-1,ab(n-1)}$, then H_0'' is rejected.

3. Test statistic

$$F_3 = \frac{SS(AB)/(a-1)(b-1)}{SSE/ab(n-1)}$$

has (under H_0''') F-distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom. If $F_3 > f_{\alpha,(a-1)(b-1),ab(n-1)}$, then H_0''' is rejected.