Journal of Computational and Applied Mathematics

Theory of J-characteristics of four-level designs under quaternary codes

--Manuscript Draft--

Manuscript Number:	ELSCAM-D-23-00096
Article Type:	Research Paper
Section/Category:	62-xx (Statistics)
Keywords:	Four-level designs; J-characteristics; Minimum G-aberration; Minimum G2-aberration; Quaternary code

Abstract:	The J-characteristics play an instrumental role in the development of generalized resolution and minimum aberration criteria for two-level designs. In this paper, the concept of J-characteristics is naturally generalized to four-level designs via quaternary codes, which maps the four-level designs to two-level designs. Based on the relationship between the minimum G2-aberration criterion of two-level design and the generalized minimum aberration criterion of its projection designs, the properties of J-characteristics of four-level designs are explored. The relationship between J-characteristics of four-level design and J-characteristics of corresponding effective two-level sub-designs is built. The generalized resolution, confounding frequency vector and B-vector of four-level design are respectively defined based on the J-characteristics and their upper bounds, and minimum G-aberration and minimum G2-aberration criteria of four-level design are proposed, which are useful to assess the goodness of four-level designs.

Professor Fatih Tank
Principal Editor of *Journal of Computational and Applied Mathematics*
Ankara University, Department of Actuarial Sciences, Ankara, Turkey

January 15, 2023

Dear Professor Fatih Tank,

Please find enclosed the manuscript:

Theory of J-characteristics for four-level designs under quaternary code
by Xiangyu Fang, Hongyi Li and Zujun Ou

to be submitted as an Original Research Article to Methodology for consideration of publication in *Journal of Computational and Applied Mathematics.* All co-authors have read and agreed with the contents of the manuscript. We certify that the submission is an original work and is not under review at any other journals.

In this manuscript, the issue of J-characteristics for four-level designs is studied. The J-characteristics play an instrumental role in the development of generalized resolution and minimum aberration criteria for two-level designs. The concept of J-characteristics is naturally generalized to four-level designs via quaternary code, which mapping the four-level designs to two-level designs. The relationship between J-characteristics of four-level designs and J-characteristics of corresponding effective two-level sub-designs is built. The generalized resolution, confounding frequency vector and B-vector of four-level design are respectively defined based on the J-characteristics and their upper bounds, and minimum G-aberration and minimum G_2-aberration criteria of four-level design are proposed, which are useful to assess the 'googness' of four-level designs.

We believe that our findings are of interest to the readers because we report here an impressive and novel direction to assess multi-level designs. The results in this paper are very important since they introduce a new direction to the theory of multi-level designs.

I appreciate for an acknowledgement receipt by mail or email. Thank you!

Sincerely yours,

Dr. Zujun Ou
Department of Statistics
College of Mathematics and Statistics

Jishou University
Jishou, Hunan Province, 416000 China
Email: ozj9325@mail.ccnu.edu.cn

Theory of J-characteristics of four-level designs under quaternary codes

Xiangyu Fang[1,2], Hongyi Li[2], Zujun Ou[2,*]

[1] *Hunan Institute of Traffic Engineering, Hengyang 421001, China*

[2] *College of Mathematics and Statistics, Jishou University, Jishou 416000, China*

Abstract

The J-characteristics play an instrumental role in the development of generalized resolution and minimum aberration criteria for two-level designs. In this paper, the concept of J-characteristics is naturally generalized to four-level designs via quaternary codes, which maps the four-level designs to two-level designs. Based on the relationship between the minimum G_2-aberration criterion of two-level design and the generalized minimum aberration criterion of its projection designs, the properties of J-characteristics of four-level designs are explored. The relationship between J-characteristics of four-level design and J-characteristics of corresponding effective two-level sub-designs is built. The generalized resolution, confounding frequency vector and B-vector of four-level design are respectively defined based on the J-characteristics and their upper bounds, and minimum G-aberration and minimum G_2-aberration criteria of four-level design are proposed, which are useful to assess the goodness of four-level designs.

MSC: 62K15, 62K99

Key words: Four-level designs; J-characteristics; Minimum G-aberration; Minimum G_2-aberration; Quaternary code.

1 Introduction

One of the important tasks in design of experiment is to find good designs and to analyze experimental data effectively, the general problem considered in this paper is how to select the 'good' fractional factorial designs. There are several optimality criteria for

*Corresponding author. E-mail: ozj9325@mail.ccnu.edu.cn (Z. J. Ou).

choosing good designs. The first is the maximum resolution criterion proposed by Box and Hunter (1961). This criterion chooses the good designs with maximum resolution, but does not distinguish among them. In situations where we have little or no knowledge about the effects that are potentially significant, the minimum aberration criterion (Fries and Hunter, 1980) has been frequently used in the selection of regular fractional factorial designs. Based on a set of J-characteristics values, Tang and Deng (1999) and Deng and Tang (1999) successively proposed generalized resolution, minimum aberration and related criteria to compare nonregular two-level designs. Subsequently, the extension version of minimum aberration named as generalized minimum aberration criterion for comparing symmetrical or asymmetrical fractional factorial designs can be found in Xu and Wu (2001) based on ANOVA model.

Obviously, the J-characteristics play a key role in the development of generalized resolution, minimum aberration criteria and their extensions for two-level designs. Furhtermore, the J-characteristics are very useful to explore the properties and construction of two-level designs, some related papers have been published after Tang and Deng (1999) and Deng and Tang (1999). Tang (2001) showed that a factorial design is uniquely determined by its J-characteristics based on the relationship between the frequency distribution of design points and its J-characteristics, which is similar to a regular factorial design uniquely determined by its defining relation. Deng and Tang (2002) classified and ranked two-level designs that were based on Hadamard matrices through J-characteristics and its related minimum aberration criteria. At the same time, a theoretical result on J-characteristics was proposed to facilitate the calculation. Evangelaras and Peveretos (2017) proposed an effective method to arrange the runs of a two-level orthogonal table into two and four blocks based on the properties of J-characteristics of orthogonal tables proposed in Deng and Tang (2002). Based on the theory of J-characteristics of two-level designs, Tang and Deng (2003) proposed a method to construct generalized minimum aberration designs of 3, 4 and 5 factors, for any run size n that is a multiple of 4. Recently, Shi and Tang (2021) undertook a comprehensive study on the construction of nonregular two-level designs with maximum generalized resolutions through J-characteristics and their lower bounds. Wang and Mee (2021) investigated the theoretical results of two-level parallel flats designs by confounding frequency vectors (Deng and Tang, 1999), which are originated from J-characteristics.

Developing a J-characteristics theory of multi-level designs is motivated by the desire of application of multi-level designs that have widely appeared in the literature. There are rarely obvious extensions of J-characteristics theory for designs with more than two levels. Bingham et al. (2009) defined the concept of J-characteristics of multi-level designs and introduced a method for constructing a rich class of orthogonal and nearly orthogonal designs that were suitable for use in computer experiments. Therefore, it is of theoretical and practical interest to further develop a general theory of J-characteristics of multi-level designs. We will focus our discussion on four-level fractional factorial designs by use of quaternary codes.

Researchers show that designs based on quaternary codes often have high efficiency, and some relevant introductions of quaternary codes are given in MacWilliams and Sloane (1977) and Wan (1997). The construction of two-level designs based on quaternary codes was firstly studied in Xu and Wong (2007). Many nonregular designs constructed by this construction method had better statistical properties than regular designs of the same size in terms of resolution, aberration and projectivity. Phoa and Xu (2009) discussed the construction and the properties of quarter-fraction nonregular designs based on quaternary codes. Soon afterwards, Zhang et al. (2011) generalized the work of Phoa and Xu (2009) to two-level (1/8)th- and (1/16)th-fractions quaternary codes designs with a trigonometric representation, and Phoa (2012) developed the basic theorems on structure and properties of such designs. Evangelaras (2015) explored the problem of constructing two-level Minimum Generalized Aberration orthogonal arrays with strength t, n runs and $q > t$ columns, using a method that employed the J-characteristics of a two-level design. Besides, uniformity of factorial designs was studied via quaternary codes, and recent results in this direction included Chatterjee et al. (2017), Hu et al. (2019), Hu et al. (2020), and Li and Qin (2020).

In this paper, based on quaternary codes, four-level designs are transformed to two-level designs, the concept of J-characteristics is naturally generalized to four-level designs. Through the set of J-characteristics, some theoretical results of four-level design are investigated from the viewpoint of projection. Moreover, the generalized resolution, confounding frequency vector and B-vector of four-level design are respectively defined based on the J-characteristics and their upper bounds, and minimum G-aberration and minimum G_2-aberration criteria of four-level design are proposed to assess the goodness of four-level designs.

The paper is organized as follows. In Section 2, some notations and preliminaries are introduced. Section 3 discusses the connection between minimum G_2-aberration and generalized minimum aberration for two-level designs. The definition of J-characteristics of four-level designs is given in Section 4, and some properties of J-characteristics are also discussed in this section. The generalized resolution, confounding frequency vector and B-vector of four-level design are respectively defined in Section 5, and minimum G-aberration and minimum G_2-aberration criteria are proposed to assess the goodness of four-level designs. Numerical examples show that these criteria are very effective for selecting better four-level designs. Finally, some concluding remarks of this paper are given in Section 6.

2 Notations and preliminaries

A symmetric U-type design with n runs and m factors of q levels, denoted by $d(n; q^m)$, is an $n \times m$ matrix, which takes entries equally often in each column from $\{-(q-1)/2, \ldots, -1, 0, 1, \ldots, (q-1)/2\}$ for odd q or from $\{-q+1, -q+3, \ldots, -1, 1, \ldots, q-1\}$ for even q. Let $\mathcal{D}(n; q^m)$ be the set of all symmetric U-type designs $d(n; q^m)$. For any design $d = (d_{i,j})_{n\times m} \in \mathcal{D}(n; q^m)$, let $d = (d_1, \ldots, d_m)$, where $d_j = (d_{1,j}, \ldots, d_{n,j})^T$ is the j-th column of d, $j = 1, \ldots, m$. Denote $Z_m = \{1, \ldots, m\}$. For any subset $u \subseteq Z_m$, let d_u be the projection design of design d on set u and $|u|$ be the cardinality of set u.

For two-level design $d \in \mathcal{D}(n; 2^m)$, the definition of J-characteristics of design d is defined in Tang (2001) as follows.

Definition 1 (Tang, 2001) For any two-level design $d \in \mathcal{D}(n; 2^m)$, d_u is the projection design of design d on set $u \subseteq Z_m$, all possible $J_u(d_u)$ values are called the J-characteristics of design d, where

$$J_u(d_u) = \sum_{i=1}^{n} \prod_{j\in u} d_{i,j}. \tag{1}$$

The above definition appears in Tang (2001) where the unique determinacy of J-characteristics to a factorial design is shown, and using this conclusion, projection justification of minimum G_2-aberration proposed in Tang and Deng (1999) is established. The above definition of J-characteristics is slightly different from original one given in Tang and Deng (1999), which is the absolute value of $J_u(d_u)$ in equation (1), i.e., $|J_u(d_u)|$. Based on the J-characteristics of design $d \in \mathcal{D}(n; 2^m)$, the B-vector is defined in Tang and Deng (1999) as follows, which aims to capture the orthogonality of design d. The minimum G_2-aberration criterion is to sequentially minimize $B_k(d)$ in Definition 2 for $k = 2, 3, \ldots, m$.

Definition 2 For any two-level design $d \in \mathcal{D}(n; 2^m)$ and $k = 2, 3, \ldots, m$, define $B_k(d) = \sum_{|u|=k} (J_u(d_u)/n)^2$, the vector $(B_2(d), \ldots, B_m(d))$ is called the B-vector of design d.

We now give the notations and preliminaries of distance distribution and generalized word-length pattern of design $d \in \mathcal{D}(n; q^m)$.

Definition 3 Suppose $d \in \mathcal{D}(n; q^m)$, $h_{ij}(d)$ is the Hamming distance between i-th and j-th rows of design d, that is, the number of positions where they differ, $i, j = 1, \ldots, n$. The vector $(E_0(d), \ldots, E_m(d))$ is called the distance distribution of design d, where

$$E_k(d) = n^{-1} \left|\{(i,j) : h_{ij}(d) = k, i, j = 1, 2, \ldots, n\}\right|, k = 0, 1, \ldots, m.$$

For any design $d \in \mathcal{D}(n; q^m)$, the MacWilliams transforms of the distance distribution are defined as

$$A_i(d) = \frac{1}{n} \sum_{j=0}^{m} P_i(j; m, q) E_j(d), i = 0, 1, \ldots, m,$$

where $P_i(j; m, q) = \sum_{r=0}^{i} (-1)^r (q-1)^{i-r} \binom{j}{r} \binom{m-j}{i-r}$ is the Krawtchouk polynomial. The vector $(A_0(d), \ldots, A_m(d))$ is called the generalized word-length pattern of design d in Xu and Wu (2001). The generalized minimum aberration criterion for selecting optimal design in $\mathcal{D}(n; q^m)$ is to sequentially minimize the generalized word-length pattern $(A_0(d), \ldots, A_m(d))$ of design d.

By the orthogonality of the Krawtchouk polynomials, it is easy to show that

$$E_j(d) = nq^{-m} \sum_{i=0}^{m} P_j(i; m, q) A_i(d). \tag{2}$$

In order to define the J-characteristics of four-level design $d \in \mathcal{D}(n; 4^m)$ and explore its properties, the replacement rule ϕ and group are respectively introduced in Definition 4 and Definition 5, which will be used in the rest of this paper.

Definition 4 For any design $d = (d_1, \ldots, d_m) \in \mathcal{D}(n; 4^m)$, the replacement rule ϕ is a rule that replaces a quaternary column of d via the following mapping by two binary columns,

$$\phi : -3 \to (-1, -1), \; -1 \to (-1, 1), \; 1 \to (1, -1), \; 3 \to (1, 1),$$

the corresponding two-level design $d' = (d'_1, d'_2, \ldots, d'_{2m-1}, d'_{2m}) \in \mathcal{D}(n; 2^{2m})$ obtained by ϕ is called the mapped design of d, and the j-th pair of columns (d'_{2j-1}, d'_{2j}) is called the mapped columns of the j-th column d_j, $j = 1, \ldots, m$.

Definition 5 Suppose G is a non-void set, and there is an algebraic operation called multiplication on G. For any two elements $a, b \in G$, the result c of the operation becomes the product of a and b, which is denoted as $c = ab$. G is said to be a group, if it still satisfies the following properties:

(i) G is closed for this multiplication.
(ii) If $a, b, c \in G$ then $(ab)c = a(bc)$.
(iii) $\forall a \in G$, $\exists\ e \in G$ such that $ea = ae = a$, e is the identity of G.
(iv) $\forall a \in G$, $\exists\ b \in G$ such that $ab = ba = e$, b becomes an inverse of a.

3 Connection between minimum G_2-aberration and generalized minimum aberration

In this section, the connection between minimum G_2-aberration and generalized minimum aberration of design $d \in \mathcal{D}(n; 2^m)$ is built, which is useful for exploration of the theory of J-characteristic of four-level design.

For any design $d \in \mathcal{D}(n; 2^m)$ and $u \subseteq Z_m$, the connection between J-characteristic $J_u(d_u)$ of design d and distance distribution of design d_u is given in Theorem 1 as follows.

Theorem 1 For any design $d \in \mathcal{D}(n; 2^m)$ and $u \subseteq Z_m$, let d_u be the projection design of design d on set $u \subseteq Z_m$, $(E_0(d_u), \ldots, E_{|u|}(d_u))$ be the distance distribution of design d_u, then

$$J_u^2(d_u) = n\sum_{j=0}^{|u|}(-1)^j E_j(d_u). \tag{3}$$

Proof. The design d_u could be rearranged into two groups by row, such that each level combination contains even entries of -1 in the first group , and each level combination contains odd entries of -1 in the second group. Suppose there are x level combinations in the first group and y level combinations in the second group. It is not hard to see that any row pair whose Hamming distance is odd only occurs between two groups of d_u, and any row pair whose Hamming distance is even only occurs within two groups of d_u. Since there are $2xy$ row pairs between two groups, which leads to $n\sum_{j \text{ is odd},\ 0\le j\le |u|} E_j(d_u) = 2xy$. Similarly, since there are $2\left(\binom{x}{2} + \binom{y}{2}\right) + (x + y) = x^2 + y^2$ row pairs in total within two groups, which leads to $n\sum_{j \text{ is even},\ 0\le j\le |u|} E_j(d_u) = x^2 + y^2$. From Definition 1, it is easy to show that $J_u(d_u) = x - y$, thus

$$\begin{aligned} J_u^2(d_u) &= x^2 + y^2 - 2xy \\ &= n\sum_{j \text{ is even},\ 0\le j\le |u|} E_j(d_u) - n\sum_{j \text{ is odd},\ 0\le j\le |u|} E_j(d_u) \end{aligned}$$

$$= \quad n\sum_{j=0}^{|u|}(-1)^j E_j(d_u),$$

which completes the proof.

For any design $d \in \mathcal{D}(n; 2^m)$ and $u \subseteq Z_m$, the connection between minimum G_2-aberration of design d and generalized minimum aberration of its projection sub-design d_u is given in Theorem 2 as follows.

Theorem 2 For any design $d \in \mathcal{D}(n; 2^m)$, let $(B_2(d), \ldots, B_m(d))$ be the B-vector of design d, $(A_0(d_u), \ldots, A_k(d_u))$ be the generalized word-length pattern of projection design d_u on $u \subseteq Z_m$ with $|u| = k$, $k = 2, \ldots, m$, then

$$B_k(d) = \frac{1}{2^k}\sum_{|u|=k}\sum_{j=0}^{k}(-1)^j\sum_{i=0}^{k}P_j(i; k, 2)A_i(d_u). \tag{4}$$

Proof. By Definition 2 and equation (3) in Theorem 1,

$$\begin{aligned} B_k(d) &= \sum_{|u|=k}\frac{J_u^2(d_u)}{n^2} \\ &= \frac{1}{n}\sum_{|u|=k}\sum_{j=0}^{k}(-1)^j E_j(d_u) \\ &= \frac{1}{2^k}\sum_{|u|=k}\sum_{j=0}^{k}(-1)^j\sum_{i=0}^{k}P_j(i; k, 2)A_i(d_u), \end{aligned}$$

which completes the proof.

An example is given to illustrate Theorem 2 in the following.

Example 1 Consider the design $\tilde{d}_1 \in \mathcal{D}(8; 2^5)$ given below.

$$\tilde{d}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}^T.$$

For $k = 2, 3, 4$, the generalized word-length patterns of all k-dimensional projection designs of design $\tilde{d}_1$ are respectively sorted in the following matrices by row,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

When $k = 5$, the 5-dimensional projection design of design $\tilde{d}_1$ is itself, and the generalized word-length pattern of design $\tilde{d}_1$ is $(1, 0, 1, 1, 0, 1)$.

From Theorem 2, the entries of B-vector are respectively obtained as $B_2(\tilde{d}_1) = 1, B_3(\tilde{d}_1) = 1, B_4(\tilde{d}_1) = 0, B_5(\tilde{d}_1) = 1$, which is the same as the ones of B-vector obtained from Definition 2 by the J-characteristics of $\tilde{d}_1$.

4 J-characteristics of four-level design

In this section, the definition of J-characteristics of four-level design $d \in \mathcal{D}(n; 4^m)$ is proposed based on its mapped design d' as given in Definition 4, which aims to depict the structure and character of four-level design d from the angle of its mapped design d'. Some properties of J-characteristics of four-level designs are investigated.

For any design $d = (d_1, \ldots, d_m) \in \mathcal{D}(n; 4^m)$, let $d' = (d'_1, d'_2, \ldots, d'_{2m-1}, d'_{2m}) \in \mathcal{D}(n; 2^{2m})$ be the mapped design of d. Due to the fact $(-3, -1, 1, 3)^T = 2(-1, -1, 1, 1)^T + (-1, 1, -1, 1)^T$, it can be easily seen that the mapped columns (d'_{2j-1}, d'_{2j}) of the j-th column d_j satisfies $d_j = 2d'_{2j-1} + d'_{2j}$, $j = 1, \ldots, m$. Therefore, the J-characteristics of four-level design $d \in \mathcal{D}(n; 4^m)$ are naturally defined as follows, which is analogous to the J-characteristics of two-level design $d \in \mathcal{D}(n; 2^m)$ in Definition 1.

Definition 6 Suppose $d \in \mathcal{D}(n; 4^m)$ and $d' = (d'_{i,j})_{n \times 2m}$ is the mapped design of design d based on replacement rule ϕ. For $i = 1, \ldots, n, j = 1, \ldots, m$, define $d'_{i,2j-1} \circledast d'_{i,2j} = 2d'_{i,2j-1} + d'_{i,2j}$. Let d_u be the projection design of design d on set $u \subseteq Z_m$, all possible $J_u(d_u)$ values are called the J-characteristics of design d, where

$$J_u(d_u) = \sum_{i=1}^{n} \prod_{j \in u} (d'_{i,2j-1} \circledast d'_{i,2j}). \tag{5}$$

For any design $d \in \mathcal{D}(n; 4^m)$, let $d' = (d'_{i,j})_{n\times 2m}$ be the mapped design of design d based on replacement rule ϕ. For any $u = \{j_1, \ldots, j_{|u|}\} \subseteq Z_m$, let $d_u = (d_{j_1}, \ldots, d_{j_{|u|}})$ be the projection design of design d on set u and $d'_u = (d'_{2j_1-1}, d'_{2j_1}, \ldots, d'_{2j_{|u|}-1}, d'_{2j_{|u|}}) \in \mathcal{D}(n; 2^{2|u|})$ be the mapped design of projection design d_u based on replacement rule ϕ, where d_{j_i} is the j_i-th column of design d and (d'_{2j_i-1}, d'_{2j_i}) is the mapped columns of d_{j_i} based on replacement rule ϕ, $i = 1, \ldots, |u|$. The $2|u|$ columns of the mapped design d'_u can be equally divided into $|u|$ groups by column and each group consists of the mapped columns (d'_{2j_i-1}, d'_{2j_i}) of the j_i-th column d_{j_i} of design d, that is, the $|u|$ groups are respectively $(d'_{2j_1-1}, d'_{2j_1}), \ldots, (d'_{2j_{|u|}-1}, d'_{2j_{|u|}})$, $i = 1, \ldots, |u|$. It is easy to show that the two columns in any group are orthogonal.

For $u = \{j_1, \ldots, j_{|u|}\} \subseteq Z_m$, let $S_u = \{(2j_i - 1, 2j_i) : i = 1, \ldots, |u|\}$ be the column index set of all groups in design d'_u. Define $W_u = \{L_k = (l_1, \ldots, l_{|u|}) : l_i \in \{2j_i - 1, 2j_i\}, i = 1, \ldots, |u|\}$ with element $L_k = (l_1, \ldots, l_{|u|})$ arranged in the lexicographic order, $k = 1, \ldots, 2^{|u|}$. The sequence of the projection designs of d'_u on $L_k = (l_1, \ldots, l_{|u|})$ is called the effective sub-design sequence of design d'_u, denoted by $s_1, \ldots, s_{2^{|u|}}$ respectively, which can be obtained by Algorithm 1 in the following. For the effective sub-design sequence $s_i = (s^i_{jk})_{n\times|u|}$ of d'_u, define the $n \times 1$ vector sequence $\mathbf{c}_i = (\prod_{k=1}^{|u|} s^i_{1k}, \ldots, \prod_{k=1}^{|u|} s^i_{nk})^T$, $i = 1, \ldots, 2^{|u|}$. Define

$$f(i) = \begin{cases} 2^{|u|}, & i = \binom{|u|}{|u|}, \\ 2^p, & \sum_{l=p+1}^{|u|} \binom{|u|}{l} < i \le \sum_{l=p}^{|u|} \binom{|u|}{l}, p = 1, \ldots, |u| - 1, \\ 2^0, & i = 2^{|u|}. \end{cases} \tag{6}$$

Based on the above notations, for any design $d \in \mathcal{D}(n; 4^m)$ and $u \subseteq Z_m$, the relationship between J-characteristics $J_u(d_u)$ of design d and J-characteristics of the effective sub-designs $s_1, \ldots, s_{2^{|u|}}$ of design d'_u is given in Theorem 3 as follows.

Theorem 3 For any design $d \in \mathcal{D}(n; 4^m)$ and $u \subseteq Z_m$, let d_u be the projection design of design d on set u, $d'_u = (d'_{i,j})_{n\times 2|u|}$ be the mapped design of design d_u based on replacement rule ϕ and $s_1, \ldots, s_{2^{|u|}}$ be the effective sub-design sequence of design d'_u, then

$$J_u^2(d_u) = \sum_{i,j=1}^{2^{|u|}} f(i)f(j)J_{Z_{|u|}}(s_i)J_{Z_{|u|}}(s_j). \tag{7}$$

Proof. For simplicity, denote the column vector $\mathbf{c}_j = (c_{1,j}, \ldots, c_{n,j})^T$ and let $J_1(\mathbf{c}_j)$ be the J-characteristic of $\mathbf{c}_j$ for $j = 1, \ldots, 2^{|u|}$. It is not hard to see that $J_{Z_{|u|}}(s_j) = J_1(\mathbf{c}_j)$.

Algorithm 1: Generation algorithm of effective sub-designs of design d'_u

---

Input: a design $d \in \mathcal{D}(n;4^m)$ and $u \subseteq Z_m$

Output: the effective sub-designs of design d'_u

```
Initialization: oddVec is the vector of odd column index, evenVec is the vector of
even column index.
for i=0 to |u| do
    Select i elements from vector evenVec in lexicographic order, which are
    stored in matrix evenMat by row.
    Obtain matrix index, which consists of the positions of the elements in
    matrix evenMat in vector evenVec.
    temp = oddVec
    for j=1 to number of rows of matrix evenMat do
        temp[index[j, ]] = evenMat[j, ]
        Output the sub-design corresponding to vector temp.
        temp = oddVec
    end
end
```

From Definition 6,

$$
\begin{aligned}
J_u^2(d_u) &= \left[\sum_{i=1}^{n}\prod_{j\in u}(d'_{i,2j-1}\circledast d'_{i,2j})\right]^2 \\
&= \left[\sum_{i=1}^{n}\prod_{j\in u}(2d'_{i,2j-1}+d'_{i,2j})\right]^2 \\
&= \Big\{2^{|u|}(c_{1,1}+\cdots+c_{n,1}) \\
&\quad +2^{|u|-1}\left[(c_{1,2}+\cdots+c_{n,2})+\cdots+\left(c_{1,\sum_{l=|u|-1}^{|u|}\binom{|u|}{l}}+\cdots+c_{n,\sum_{l=|u|-1}^{|u|}\binom{|u|}{l}}\right)\right] \\
&\quad +\cdots+2^{0}(c_{1,2^{|u|}}+\cdots+c_{n,2^{|u|}})\Big\}^2 \\
&= \left[f(1)J_1(\mathbf{c}_1)+f(2)J_1(\mathbf{c}_2)+\cdots+f(2^{|u|})J_1(\mathbf{c}_{2^{|u|}})\right]^2 \\
&= \sum_{i,j=1}^{2^{|u|}} f(i)f(j)J_1(\mathbf{c}_i)J_1(\mathbf{c}_j) \\
&= \sum_{i,j=1}^{2^{|u|}} f(i)f(j)J_{Z_{|u|}}(s_i)J_{Z_{|u|}}(s_j),
\end{aligned}
$$

which completes the proof.

For any design $d \in \mathcal{D}(n;4^m)$ and $u \subseteq Z_m$, the relationship between J-characteristic

$J_u(d_u)$ of design d and the distance distributions of the effective sub-design sequence of design d'_u is given in Theorem 4 as follows.

Theorem 4 For any design $d \in \mathcal{D}(n; 4^m)$ and $u \subseteq Z_m$, let d_u be the projection design of design d on set u, $d'_u = (d'_{i,j})_{n\times 2|u|}$ be the mapped design of design d_u based on replacement rule ϕ, $s_1, \ldots, s_{2^{|u|}}$ be the effective sub-design sequence of design d'_u and $(E_0(s_i), \ldots, E_{|u|}(s_i))$ be the distance distribution of d_i, $i = 1, \ldots, 2^{|u|}$, then

$$J_u^2(d_u) = n \sum_{i,j=1}^{2^{|u|}} f(i)f(j)\sqrt{\sum_{k,l=0}^{|u|}(-1)^{k+l}E_k(s_i)E_l(s_j)}. \tag{8}$$

Proof. From Theorem 1 and Theorem 3,

$$\begin{aligned}
J_u^2(d_u) &= \sum_{i,j=1}^{2^{|u|}} f(i)f(j)J_{Z_{|u|}}(s_i)J_{Z_{|u|}}(s_j) \\
&= n\sum_{i,j=1}^{2^{|u|}} f(i)f(j)\sqrt{\sum_{k=0}^{|u|}(-1)^k E_k(s_i)\sum_{l=0}^{|u|}(-1)^l E_l(s_j)} \\
&= n\sum_{i,j=1}^{2^{|u|}} f(i)f(j)\sqrt{\sum_{k,l=0}^{|u|}(-1)^{k+l}E_k(s_i)E_l(s_j)},
\end{aligned}$$

which completes the proof.

An example is given to illustrate Theorem 3 and Theorem 4 in the following.

Example 2 Consider the four-level design $\tilde{d}_2 \in \mathcal{D}(8; 4^5)$ given below.

$$\tilde{d}_2 = \begin{bmatrix} -3 & 1 & 3 & -1 & -1 & -3 & 1 & 3 \\ 3 & -3 & -1 & 1 & -3 & -1 & 1 & 3 \\ -1 & 1 & -3 & -3 & -1 & 3 & 3 & 1 \\ 1 & -3 & 1 & -3 & 3 & -1 & 3 & -1 \\ -3 & -3 & -1 & 1 & 3 & 1 & -1 & 3 \end{bmatrix}^T.$$

The two-level mapped design $\tilde{d}'_2 \in \mathcal{D}(8; 2^{10})$ of $\tilde{d}_2$ based on replacement rule ϕ is given as follows

$$\tilde{d}_2' = \begin{bmatrix} -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

To save space, only the numerical results of the case $u = Z_5$ are given. From Definition 6, the J-characteristic $J_{Z_5}(\tilde{d}_2) = 144$. From Definition 1, the J-characteristics of all designs in the effective sub-design sequence of design $\tilde{d}_2'$ are obtained, which are listed in the following vector in order,

$$(-8, 0, 0, 0, 0, 0, 0, 0, 8, 0, 8, -8, 0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -8, 0, 8, -8, 0, 0).$$

Based on the above J-characteristics and Theorem 3, $J^2_{Z_5}(\tilde{d}_2) = 20736$.

On the other hand, the distance distributions of all designs in the effective sub-design sequence of design $\tilde{d}_2'$ are obtained from Definition 3, which are stored in the following 32×6 matrix by row

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&2&2&2&2&2&2&3&1&3&1&2&1&3&1&3&1&2&1&2&2&2&1&2&1&2&1&2&1&1&2&2 \\ 0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}^T.$$

It is easy to show that $J^2_{Z_5}(\tilde{d}_2) = 20736$ from Theorem 4. Thus, both the J-characteristic $J_{Z_5}(\tilde{d}_2)$ from Theorem 3 and the one from Theorem 4 are the same as the results calculated by Definition 6.

5 Generalized resolution and minimum aberration criteria of four-level designs

Given an orthogonal design $OA(n; 2^m)$, Deng and Tang (1999) proved that all J-characteristics are multiples of 4. For any design $d \in \mathcal{D}(n; 4^m)$, the mapped design of design d based on replacement rule ϕ is not necessarily an orthogonal design $OA(n; 2^{2m})$. In order to investigate the properties of J-characteristics of design $d \in \mathcal{D}(n; 4^m)$, it is bringing the attention to the case of n is a multiple of 4 in the following.

Define $\mathcal{E} = \{4i : i \text{ is positive integer}\}$, $\mathcal{F} = \{i : i \text{ is even}\}$, $\mathcal{V} = \{\mathbf{a} = (a_1, \ldots, a_n)^T : a_i \in \{1, -1\}, n \in \mathcal{E}, \sum_{i=1}^n a_i \in \mathcal{F}\}$. Define a new operation $\circ$ on the set $\mathcal{V}$: $\forall \mathbf{a}, \mathbf{b} \in \mathcal{V}$, where $\mathbf{a} = (a_1, \ldots, a_n)^T, \mathbf{b} = (b_1, \ldots, b_n)^T$, then $\mathbf{a} \circ \mathbf{b} = (a_1 b_1, \ldots, a_n b_n)^T$. Lemma 1 shows that the set $\mathcal{V}$ is a group for the operation $\circ$ as follows.

Lemma 1 The set $\mathcal{V}$ is a group for the operation $\circ$.

Proof. The four conditions of $\mathcal{V}$ as a group given in Definition 5 are verified respectively.

(i) $\forall\ \mathbf{a}, \mathbf{b} \in \mathcal{V}$, $\sum_{i=1}^n a_i, \sum_{i=1}^n b_i \in \mathcal{F}$, suppose $\mathbf{a}$ and $\mathbf{b}$ respectively contain $2k_1$ and $2k_2$ -1's, where $0 \le k_1, k_2 \le n/2$. Denote $\mathbf{c} = \mathbf{a} \circ \mathbf{b}$, $(\mathbf{a}, \mathbf{b})$ is the $n \times 2$ matrix consisted of $\mathbf{a}$ and $\mathbf{b}$. If the number of pair $(1, 1)$ in $(\mathbf{a}, \mathbf{b})$ is k , then the numbers of pairs $(1, -1)$ and $(-1, 1)$ in $(\mathbf{a}, \mathbf{b})$ are $n - 2k_1 - k$ and $n - 2k_2 - k$, respectively. Therefore, the number of -1's in the vector $\mathbf{c}$ is $(n - 2k_1 - k) + (n - 2k_2 - k) = 2(n - k_1 - k_2 - k)$, thus $\sum_{i=1}^n c_i \in \mathcal{F}$, i.e., $\mathbf{c} \in \mathcal{V}$.

(ii) $\forall\ \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{V}$, $(\mathbf{a} \circ \mathbf{b}) \circ \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \circ \mathbf{c}) = (a_1 b_1 c_1, \ldots, a_n b_n c_n)^T$.

(iii) For $\mathbf{e} = (1, 1, \ldots, 1)^T \in \mathcal{V}$ and $\forall\ \mathbf{a} \in \mathcal{V}$, $\mathbf{e} \circ \mathbf{a} = \mathbf{a} \circ \mathbf{e} = \mathbf{a}$. Thus, $\mathbf{e}$ is the unity element of $\mathcal{V}$.

(iv) $\forall\ \mathbf{a} \in \mathcal{V}$, $\mathbf{a} \circ \mathbf{a} = \mathbf{e}$, thus the inverse of $\mathbf{a}$ is itself.

Propostion 1 For any design $d' \in \mathcal{D}(n; 2^m)$ with $n \in \mathcal{E}$ and $u \subseteq Z_m$, let d'_u be the projection design of design d' on set u, then $J_u(d'_u)$ is a multiple of 4.

Proof. $\forall u = \{i_1, \ldots, i_k\} \subseteq Z_m$, where $|u| = k, k = 0, 1, \ldots, m$, denote the projection design of design d' on set u as $d'_u = (d_{i_1}, \ldots, d_{i_k})$. Obviously, $d_{i_j} \in \mathcal{V}$, $j = 1, \ldots, k$. Denote $\mathbf{c} = d_{i_1} \circ \cdots \circ d_{i_k}$, by the closeness of group $\mathcal{V}$, then vector $\mathbf{c} \in \mathcal{V}$, i.e. there are even number of -1's in vector $\mathbf{c}$. Let x be the number of -1's in vector $\mathbf{c}$, then the number of 1's in vector $\mathbf{c}$ is $n - x$. Thus, $J_u(d'_u) = J_1(\mathbf{c}) = n - 2x$ is a multiple of 4 since $n \in \mathcal{E}$, which completes the proof.

For any design $d \in \mathcal{D}(n; 4^m)$, it is obvious that n is a multiple of 4 since d is a U-type design. Therefore, for any $u \subseteq Z_m$, the run number of the projection design d_u of design d on set u is a multiple of 4, and the same as for the mapped design d'_u of d_u. From Propostion 1, the following result is obvious.

Propostion 2 For any design $d \in \mathcal{D}(n; 4^m)$ and $u \subseteq Z_m$, let d_u be the projection design of design d on set u and $J_u(d_u)$ be its J-characteristic. Then $J_u(d_u)$ is a multiple of 4.

In order to give an upper bound of the J-characteristics $J_u(d_u)$ of four-level design $d \in \mathcal{D}(n; 4^m)$, two lemmas are required.

Define $\mathcal{V}_1 = \{\mathbf{a} = (a_1, \dots, a_n)^T, a_i \in \{1, -1\}, n \in \mathcal{E}, \sum_{i=1}^n a_i = 0\}$ and $\mathcal{V}_2 = \{\mathbf{a} = (a_1, \dots, a_n)^T, a_i \in \{1, -1\}, n \in \mathcal{E}, \sum_{i=1}^n a_i \in \mathcal{E}\}$. Obviously, both $\mathcal{V}_1$ and $\mathcal{V}_2$ are subsets of set $\mathcal{V}$, i.e., $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$.

Lemma 2 For any $\mathbf{a}_1 \in \mathcal{V}_1, \mathbf{a}_2 \in \mathcal{V}$, let γ be the number of pair (1, 1) in the $n \times 2$ matrix $(\mathbf{a}_1, \mathbf{a}_2)$, $J_2(\mathbf{a}_1, \mathbf{a}_2)$ and $J_1(\mathbf{a}_2)$ are respectively the J-characteristics of $(\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{a}_2$. Then $J_2(\mathbf{a}_1, \mathbf{a}_2) = 4\gamma - J_1(\mathbf{a}_2) - n$, where $0 \leq \gamma \leq n/2$.

Proof. Since $\mathbf{a}_2 \in \mathcal{V}$, the numbers of 1 and -1 in $\mathbf{a}_2$ are respectively $(n+J_1(\mathbf{a}_2))/2$ and $(n - J_1(\mathbf{a}_2))/2$. It is noting that γ is the number of pair (1, 1) in matrix $(\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{a}_1 \in \mathcal{V}_1$, the pair $(1, -1)$ appears $n/2-\gamma$ times in matrix $(\mathbf{a}_1, \mathbf{a}_2)$. Furthermore, the pair $(-1, 1)$ appears $(n+J_1(\mathbf{a}_2))/2-\gamma$ times in matrix $(\mathbf{a}_1, \mathbf{a}_2)$, and the pair $(-1, -1)$ appears $\gamma - J_1(\mathbf{a}_2)/2$ times in matrix $(\mathbf{a}_1, \mathbf{a}_2)$. From Definition 1, $J_2(\mathbf{a}_1, \mathbf{a}_2) = 4\gamma - J_1(\mathbf{a}_2) - n$, which completes the proof.

Lemma 3 For any $\mathbf{a}_1, \mathbf{a}_2 \in \mathcal{V}_1, \mathbf{a}_3 \in \mathcal{V}_2$, let $J_2(\mathbf{a}_2, \mathbf{a}_3)$ and $J_2(\mathbf{a}_1, \mathbf{a}_3)$ be the J-characteristics of $(\mathbf{a}_2, \mathbf{a}_3)$ and $(\mathbf{a}_1, \mathbf{a}_3)$, respectively. If $\mathbf{a}_1^T\mathbf{a}_2 = 0$, then $J_2(\mathbf{a}_2, \mathbf{a}_3) + J_2(\mathbf{a}_1, \mathbf{a}_3) \leq n$.

Proof. For any $\mathbf{a}_1 \in \mathcal{V}_1$, the entries of $\mathbf{a}_1$ are divided into two groups, where the entries of the first group are all 1's and the entries of the second group are all -1's. Let γ be the number of pair (1, 1) in matrix $(\mathbf{a}_1, \mathbf{a}_3)$, and $J_1(\mathbf{a}_3)$ be the J-characteristic of $\mathbf{a}_3$. From Lemma 2, $J_2(\mathbf{a}_1, \mathbf{a}_3) = 4\gamma - J_1(\mathbf{a}_3) - n$.

If $\mathbf{a}_1^T\mathbf{a}_2 = 0$, $J_1(\mathbf{a}_3)$ is a positive integer and it is a multiple of 4, there are at most $n/4$ entries in $\mathbf{a}_2$ corresponding to the first group of $\mathbf{a}_1$ that can form the pair (1, 1) coupling with the entries of $\mathbf{a}_3$, and there are at most $\min\{(n + J_1(\mathbf{a}_3))/2 - \gamma, n/4\}$ entries in $\mathbf{a}_2$ corresponding to the second group of $\mathbf{a}_1$ that can form the pair (1, 1) coupling with the entries of $\mathbf{a}_3$. Thus, the number of pair (1, 1) in matrix $(\mathbf{a}_2, \mathbf{a}_3)$ is at most $n/4 + \min\{(n + J_1(\mathbf{c}_3))/2 - \gamma, n/4\}$. It is easy to show that if and only if $(n + J_1(\mathbf{c}_3))/2 - \gamma = n/4$, $J_1(\mathbf{a}_3)$ is the minimum such that the pair (1, 1) appears the maximum times $n/2$ in matrix $(\mathbf{a}_2, \mathbf{a}_3)$.

Combining the two conditions obtained above, i.e.,

$$\begin{cases} J_2(\mathbf{a}_1, \mathbf{a}_3) = 4\gamma - J_1(\mathbf{a}_3) - n, \\ \frac{n+J_1(\mathbf{a}_3)}{2} - \gamma = \frac{n}{4}. \end{cases}$$

Thus, $J_1(\mathbf{a}_3) = J_2(\mathbf{a}_1, \mathbf{a}_3)$. On the other hand, from Lemma 2, the maximum J-characteristic of matrix $(\mathbf{a}_2, \mathbf{a}_3)$ is $4n/2 - J_1(\mathbf{a}_3) - n = n - J_2(\mathbf{a}_1, \mathbf{a}_3)$, i.e., $J_2(\mathbf{a}_2, \mathbf{a}_3) \leq$

$n - J_2(\mathbf{a}_1, \mathbf{a}_3)$ implies $J_2(\mathbf{a}_2, \mathbf{a}_3) + J_2(\mathbf{a}_1, \mathbf{a}_3) \le n$, which completes the proof.

Based on Lemma 3, an upper bound of the J-characteristic $J_u(d_u)$ for any four-level design $d \in \mathcal{D}(n; 4^m)$ is given in Theorem 5 as follows.

Theorem 5 For any design $d \in \mathcal{D}(n; 4^m)$ and $u \subseteq Z_m$ with $|u| = k$, $k = 1, \ldots, m$, let d_u be the projection design of design d on the set u and $J_u(d_u)$ be its J-characteristic. Then $J_u(d_u) \le n(3^k + 1)/2$.

Proof. Let d'_u be the mapped design of design d_u via replacement rule ϕ, and d'_0 be the first design in the effective sub-design sequence of design d'_u. Denote $l = J_{Z_k}(d'_0)$, from Propostion 1, l is a multiple of 4 and can be expressed as $l = 4t, t = 0, 1, \ldots, n/4$. From Lemma 3, when k is even,

$$\begin{aligned}
J_u(d_u) &\le l\binom{k}{k}2^k + (n-l)\binom{k}{k-1}2^{k-1} + l\binom{k}{k-2}2^{k-2} + \cdots + l\binom{k}{0}2^0 \\
&= n\left[\binom{k}{k-1}2^{k-1} + \binom{k}{k-3}2^{k-3} + \cdots + \binom{k}{1}2^1\right] + l\sum_{i=0}^{k}\binom{k}{i}(-2)^i \\
&= \frac{n[3^k - (-1)^k]}{2} + l(-1)^k \\
&= \frac{n(3^k - 1)}{2} + l \\
&\le \frac{n(3^k + 1)}{2}.
\end{aligned}$$

Similarly, when k is odd,

$$\begin{aligned}
J_u(d_u) &\le l\binom{k}{k}2^k + (n-l)\binom{k}{k-1}2^{k-1} + l\binom{k}{k-2}2^{k-2} + \cdots + (n-l)\binom{k}{0}2^0 \\
&= n\left[\binom{k}{k-1}2^{k-1} + \binom{k}{k-3}2^{k-3} + \cdots + \binom{k}{0}2^0\right] - l\sum_{i=0}^{k}\binom{k}{i}(-2)^i \\
&= \frac{n[3^k + (-1)^k]}{2} - l(-1)^k \\
&= \frac{n(3^k - 1)}{2} + l \\
&\le \frac{n(3^k + 1)}{2},
\end{aligned}$$

which completes the proof.

For any design $d \in \mathcal{D}(n; 4^m)$, let r be the smallest integer such that $\max_{|u|=r} |J_u(d_u)| > 0$, where the maximization is taken over all the subsets u of size r. Then the generalized resolution of design $d \in \mathcal{D}(n;4^m)$ is defined as $R(d) = r + \eta$, where

$$\eta = 1 - 2\max_{|u|=r} |J_u(d_u)|/[n(3^r+1)].$$

For any design $d \in \mathcal{D}(n;4^m)$ and $u \subseteq Z_m$ with $|u| = k$, $k = 1,\ldots,m$, let f_{kj} be the frequency of projection design d_u such that $|u| = k$ and $|J_u(d_u)| = 4(t_k + 1 - j)$, where $t_k = n(3^k+1)/8$ and $j = 1,\ldots,t_k$. The confounding frequency vector $F(d)$ of design d is defined as the following vector of length $\sum_{k=2}^m t_k$

$$F(d) = [(f_{21},\ldots,f_{2t_2});(f_{31},\ldots,f_{3t_3});\ldots;(f_{m1},\ldots,f_{mt_m})].$$

For any design $d \in \mathcal{D}(n;4^m)$ and $u \subseteq Z_m$ with $|u| = k$, $k = 2,\ldots,m$, $B(d) = (B_2(d),\ldots,B_m(d))$ is defined as the B-vector of design $d \in \mathcal{D}(n;4^m)$, where

$$B_k(d) = \sum_{|u|=k} \beta_k^2(d_u)$$

and $\beta_k(d_u) = 2|J_u(d_u)|/[n(3^k+1)]$ are normalized J-characteristics. For $k = 2,\ldots,m$, it is easy to show that

$$B_k(d) = \sum_{j=1}^{t_k} f_{kj}\left[\frac{8(t_k+1-j)}{n(3^k+1)}\right]^2.$$

Based on the confounding frequency vector $F(d)$ and B-vector $B(d)$ of design d defined above, the minimum G-aberration criterion and the minimum G_2-aberration criterion for ranking four-level designs in $\mathcal{D}(n;4^m)$ are respectively defined as follows.

Definition 7 For any two designs $\tilde{d}_1, \tilde{d}_2 \in \mathcal{D}(n;4^m)$, let $f_r(\tilde{d}_1)$ and $f_r(\tilde{d}_2)$ be the the r-th entries of confounding frequency vectors $F(\tilde{d}_1)$ and $F(\tilde{d}_2)$, respectively, $r = 1,2,\ldots,\sum_{k=2}^m t_k$. Then design $\tilde{d}_1$ is said to have less G-aberration if $f_r(\tilde{d}_1) < f_r(\tilde{d}_2)$, and $f_i(\tilde{d}_1) = f_i(\tilde{d}_2), i = 1,\ldots,r-1$. If no design has less G-aberration than design $\tilde{d}_1$, then design $\tilde{d}_1$ is said to have minimum G-aberration.

Definition 8 For any two designs $\tilde{d}_1, \tilde{d}_2 \in \mathcal{D}(n;4^m)$, let r be the the smallest integer such that $B_r(\tilde{d}_1) \neq B_r(\tilde{d}_2)$. Then design $\tilde{d}_1$ is said to have less G_2-aberration if $B_r(\tilde{d}_1) < B_r(\tilde{d}_2)$, and $B_i(\tilde{d}_1) = B_i(\tilde{d}_2), i = 1,\ldots,r-1$. If no design has less G_2-aberration than design $\tilde{d}_1$, then design $\tilde{d}_1$ is said to have minimum G_2-aberration.

Finally, three examples are given to illustrate above defined criteria for ranking four-level designs.

Example 3 Consider two designs $\tilde{d}_3 \in \mathcal{D}(8; 4^8)$ and $\tilde{d}_4 \in \mathcal{D}(8; 4^8)$ given Table 1.

Table 1: Four-level designs $\tilde{d}_3 \in \mathcal{D}(8; 4^8), \tilde{d}_4 \in \mathcal{D}(8; 4^8)$

$\tilde{d}_3$								$\tilde{d}_4$							
3	-3	-1	3	3	1	-3	-1	-3	3	-1	-3	-1	1	-3	1
1	-1	3	-3	-3	1	3	3	1	3	-1	3	1	1	3	-3
-1	-3	3	1	-1	-3	-3	-3	-3	-1	1	1	3	-3	3	1
-3	3	1	-3	-3	-3	1	-1	-1	-3	-3	1	-3	3	1	-1
-3	1	-3	3	-1	3	-1	3	3	1	3	-1	1	3	1	3
1	1	1	1	3	-1	-1	1	1	-3	1	3	-1	-1	-3	3
-1	3	-1	-1	1	3	3	1	-1	1	3	-1	-3	-3	-1	-3
3	-1	-3	-1	1	-1	1	-3	3	-1	-3	-3	3	-1	-1	-1

The generalized resolutions of $\tilde{d}_3$ and $\tilde{d}_4$ are respectively 2.2 and 2.6, thus design $\tilde{d}_4$ has higher generalized resolution than design $\tilde{d}_3$.

The confounding frequency vectors of $\tilde{d}_3$ and $\tilde{d}_4$ are respectively $F(\tilde{d}_3) = [(0, 0, 1, 1, 4, 1, 6, 3, 5, 3); (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 1, 0, 0, 0, 8, 2, 0, 3, 8, 3, 5, 1, 13, 7); \ldots]$ and $F(\tilde{d}_4) = [(0, 0, 0, 0, 0, 0, 2, 7, 4, 9); (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 4, 0, 1, 1, 5, 1, 2, 5, 4, 0, 6, 1, 6, 11, 1, 3); \ldots]$, thus design $\tilde{d}_4$ has less G-aberration than design $\tilde{d}_3$.

The B-vectors of $\tilde{d}_3$ and $\tilde{d}_4$ are respectively $B(\tilde{d}_3) = (4.28, 3.48, 3.33, 0.80, 0.17, 0.02, 0.00)$ and $B(\tilde{d}_4) = (1.20, 4.92, 1.40, 0.36, 0.05, 0.00, 0.00)$, thus design $\tilde{d}_4$ has less G_2-aberration than design $\tilde{d}_3$.

Example 4 Consider two designs $\tilde{d}_5 \in \mathcal{D}(8; 4^3)$ and $\tilde{d}_6 \in \mathcal{D}(8; 4^3)$ given in Table 2, where design $\tilde{d}_6$ is obtained by level permutation $(-3, -1, 1, 3) \to (1, -1, -3, 3)$ of design $\tilde{d}_5$ in the third column.

The generalized resolutions of $\tilde{d}_5$ and $\tilde{d}_6$ are respectively 2.6 and 2.2, thus design $\tilde{d}_5$ has higher generalized resolution than design $\tilde{d}_6$.

The confounding frequency vectors of $\tilde{d}_5$ and $\tilde{d}_6$ are respectively $F(\tilde{d}_5) = [(0, 0, 0, 0, 0, 0, 1, 0, 0, 0); (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)]$, $F(\tilde{d}_6) = [(0, 0, 1, 0, 0, 0, 2, 0, 0, 0); (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)]$, thus design $\tilde{d}_5$ has less G-aberration than design $\tilde{d}_6$.

The B-vectors of designs $\tilde{d}_5$ and $\tilde{d}_6$ are respectively $B(\tilde{d}_5) = (0.16, 0.02)$ and $B(\tilde{d}_6) = (0.96, 0.02)$, thus design $\tilde{d}_5$ has less G_2-aberration than design $\tilde{d}_6$.

It is noting that the generalized word-length patterns of both designs $\tilde{d}_5$ and $\tilde{d}_6$ are $(0, 3, 4)$, they can be further ranked by the criteria in this paper.

Table 2: Four-level designs $\tilde{d}_5 \in \mathcal{D}(8; 4^3), \tilde{d}_6 \in \mathcal{D}(8; 4^3)$

$\tilde{d}_5$			$\tilde{d}_6$		
3	1	1	3	1	-3
3	3	-1	3	3	-1
-1	1	-1	-1	1	-1
-1	3	1	-1	3	-3
1	-3	-3	1	-3	1
-3	-1	-3	-3	-1	1
1	-1	3	1	-1	3
-3	-3	3	-3	-3	3

Example 5 Consider two designs $\tilde{d}_7 \in \mathcal{D}(8; 4^7)$ and $\tilde{d}_8 \in \mathcal{D}(8; 4^7)$ given in Table 3.

Table 3: Four-level designs $\tilde{d}_7 \in \mathcal{D}(8; 4^7), \tilde{d}_8 \in \mathcal{D}(8; 4^7)$

$\tilde{d}_7$							$\tilde{d}_8$						
-3	3	3	-3	1	1	-3	-3	1	3	1	-1	-1	-1
-1	-1	-3	3	-1	1	1	1	-3	1	-3	1	3	1
3	1	1	-3	-3	-1	1	3	1	-3	3	-3	3	-3
3	-3	3	1	-1	-3	-1	-1	3	1	-1	3	-1	-3
1	-3	-1	3	3	-1	-3	-1	-1	3	3	-1	-3	1
-1	3	-1	-1	-3	3	-1	3	-1	-1	-3	3	1	-1
1	1	-3	-1	1	-3	3	1	3	-1	1	-3	-3	3
-3	-1	1	1	3	3	3	-3	-3	-3	-1	1	1	3

The generalized resolutions of both $\tilde{d}_7$ and $\tilde{d}_8$ are 2.2, that is, the generalized resolution of design $\tilde{d}_7$ is the same as the one of design $\tilde{d}_8$.

The confounding frequency vectors of $\tilde{d}_7$ and $\tilde{d}_8$ are respectively $F(\tilde{d}_7) = [(0, 0, 2, 0, 0, 0, 7, 0, 0, 0); (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 1, 0, 5, 0, 0, 0, 10, 0, 2, 0, 5, 0, 4, 0); \ldots]$, $F(\tilde{d}_8) = [(0, 0, 1, 0, 0, 2, 5, 3, 2, 8); (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 2, 1, 4, 1, 3, 2, 1, 3, 5, 2, 0, 2, 3, 3); \ldots]$, thus design $\tilde{d}_8$ has less G-aberration than design $\tilde{d}_7$.

The B-vectors of designs $\tilde{d}_7$ and $\tilde{d}_8$ are respectively $B(\tilde{d}_7) = (2.40, 2.85, 0.86, 0.21, 0.05, 0.00)$ and $B(\tilde{d}_8) = (2.37, 3.49, 1.43, 0.60, 0.16, 0.00)$, thus design $\tilde{d}_8$ has less G_2-aberration than design $\tilde{d}_7$.

It can be seen that although the generalized resolutions of designs $\tilde{d}_7$ and $\tilde{d}_8$ are the same, they can be ranked by minimum G-aberration and minimum G_2-aberration criteria.

6 Concluding remarks

It is much easier to study and deal with two-level designs compared with multi-level designs as the former have more comprehensible theoretical system. Therefore, it is desirable to obtain relevant theories of J-characteristics of four-level designs based on theories of J-characteristics of two-level designs.

Focusing on the discussion of four-level designs in this paper, the theory of J-characteristics of four-level designs is studied based their mapped two-level designs, which are obtained from quaternary codes. Firstly, the relationship between the minimum G_2-aberration criterion of two-level design and the generalized minimum aberration criterion of its projection designs is given. Subsequently, the definition of J-characteristics of four-level designs is proposed via the bridge of quaternary codes, and the connection between J-characteristics of four-level design and J-characteristics of effective two-level sub-designs is built. Finally, generalized resolution, minimum G-aberration and minimum G_2-aberration criteria for four-level designs are also given based on the J-characteristics of four-level design, which play an important role in screening four-level designs with less aberration.

A question arises now: How to construct a class of optimal four-level designs with minimum G_2-aberration? It is a potential work worthy for further investigations in the future.

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant Nos. 11961027, 12161040, 11701213), Hunan Provincial Natural Science Foundation of China (Grant Nos. 2021JJ30550, 2020JJ4497) and Scientific Research Plan Item of Hunan Provincial Department of Education (Grant No. 22A0355).

References

Bingham, D., Sitter, R. R. and Tang, B. (2009). Orthogonal and nearly orthogonal designs for computer experiments. *Biometrika*, **96(1)**: 51–65.

Box, G. E. P. and Hunter, J. S. (1961). The 2^{k-p} fractional factorial designs. *Technometrics*, **3**: 311–351 and 449–458.

Chatterjee, K., Ou, Z. J., Phoa, F. K. H. and Qin, H. (2017). Uniform four-level designs from two-level designs: a new look. *Statistica Sinica*, **27(1)**: 171–186.

Deng, L. Y. and Tang, B. (1999). Generalized resolution and minimum aberration criteria for Plackett-Burman and other nonregular factorial designs. *Statistica Sinica*, **9(4)**: 1071–1082.

Deng, L. Y. and Tang, B. (2002). Design selection and classification for Hadamard matrices using generalized minimum aberration criteria. *Technometrics*, **44(2)**: 173–184.

Evangelaras, H. (2015). Construction of minimum generalized aberration two-level orthogonal arrays. *Electronic Journal of Statistics*, **9(2)**: 2689–2705.

Evangelaras, H. and Peveretos, C. (2017). Efficient arrangements of two-level orthogonal arrays in two and four blocks. *Statistics*, **51(6)**: 1326–1341.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics*, **22**: 601–608.

Hu, L. P., Li, H. Y. and Ou, Z. J. (2019). Constructing optimal four-level designs via Gray map code. *Metrika*, **82(5)**: 573–587.

Hu, L. P., Ou, Z. J. and Li, H. Y. (2020). Construction of four-level and mixed-level designs with zero *Lee* discrepancy. *Metrika*, **83(1)**: 129–139.

Li, H. Y. and Qin, H. (2020). New lower bounds of four-level and two-level designs via two transformations. *Statistical Papers*, **61(3)**: 1231–1243.

MacWilliams, F. J. and Sloane, N. J. A. (1977). The Theory of Error Correcting Codes. *Amsterdam: North-Holland.*

Phoa, F. K. H. (2012). A code arithmetic approach for quaternary code designs and its application to (1/64)th-fractions. *Annals of Statistics*, **40(6)**: 3161–3175.

Phoa, F. K. H. and Xu, H. (2009). Quarter-fraction factorial designs constructed via quaternary codes. *Annals of Statistics*, **37(5A)**: 2561–2581.

Shi, C. and Tang, B. (2021). On construction of nonregular two-level factorial designs with maximum generalized resolutions. *Statistica Sinica*, Doi: 10.5705/ss.202021.0024.

Tang, B. (2001). Theory of J-characteristics for fractional factorial designs and projection justification of minimum G_2-aberration. *Biometrika*, **88(2)**: 401–407.

Tang, B. and Deng, L. Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Annals of Statistics*, **27(6)**: 1914–1926.

Tang, B. and Deng, L. Y. (2003). Construction of generalized minimum aberration designs of 3, 4 and 5 factors. *Journal of Statistical Planning and Inference*, **113(1)**: 335–340.

Wang, C. and Mee, R. W. (2021). Two-level parallel flats designs. *Annals of Statistics*, **49(5)**: 3015–3042.

Wan, Z. X. (1997). Quaternary Codes. *Singapore: World Scientific.*

Xu, H. and Wong, A. (2007). Two-level nonregular designs from quaternary linear codes. *Statistica Sinica*, **17(3)**: 1191–1213.

Xu, H. and Wu, C. F. J. (2001). Generalized minimum aberration for asymmetrical fractional factorial designs. *Annals of Statistics*, **29(2)**: 549–560.

Zhang, R., Phoa, F. K. H., Mukerjee, R. and Xu, H. (2011). A trigonometric approach to quaternary code designs with application to one-eighth and one-sixteenth fractions. *Annals of Statistics*, **39(2)**: 931–955.