Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Processes**

Academic contact during examination: Andrea Riebler
Phone: 4568 9592

Examination date: December 16th 2013
Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College or HP30S.
- Statistiske tabeller og formler, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten formulas and notes.

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English
Number of pages: 6
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

Consider the Markov chain whose transition probability matrix is given by

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

a)
- Draw a state transition diagram and determine the equivalence classes.
- Which states are recurrent and which states are transient? Justify your answer.
- Calculate the following probabilities:

$$P(X_4 = 3 \mid X_3 = 1, X_2 = 2) \quad \text{and} \quad P(X_2 = 3 \mid X_0 = 1)$$

b)
- Compute the probability of absorption into state 0 starting from state 1.
- Starting in state 1, compute the expected time spent in each of states 1 and 2 prior to absorption in state 0 or 3.

Problem 2

Let $\{X_n, n = 0, 1, \ldots\}$ denote a branching process in which all individuals are assumed to have offsprings independently of each other. X_n denotes the population size at the n-th generation and we assume that $X_0 = 1$. By the end of its life time, each individual is assumed to have produced no offspring with probability $P_0 = \frac{1}{8}$, one offspring with probability $P_1 = \frac{1}{2}$ and two offspring with probability $P_2 = \frac{3}{8}$.

a)
- Explain why this process is a Markov chain.
- Derive the state space. Which states are transient and which states are recurrent?

b) Compute the expected number of offsprings of a single individual. What is the probability that the population will die out?

Problem 3

An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 6$ claims per week. Let $N(t)$ be the number of insurance claims at time t (measured in weeks) and assume that $N(0) = 0$.

a)
- What is the expected time until the fifteenth insurance claim is paid?
- Find $E(N(4) - N(2) \mid N(1) = 5)$
- Compute also $P(N(3) \geq 12)$.

Assume that the amount of money paid on each policy are independent exponentially distributed random variables with common mean 12000 kroner. Assume also that the amount of money paid on each policy is independent of the number of claims that are paid out.

b) What is the expected value and variance of the total amount of money paid by the insurance company in a four-week span?

Problem 4

Biathlon commonly refers to the winter sport that combines cross-country skiing and rifle shooting. Assume that the inhabitants of Oslo want to improve their Biathlon skills and go to a popular skiing area to train. There is a stadium with three public shooting stands available. Skiers arrive at the shooting stands according to a Poisson process with rate 5 skiers per minute, i.e. $\lambda = 1/12$ skier per second. If a shooting stand is free an entering skier immediately starts to shoot and then leaves directly the stadium. If all stands are occupied he waits in line and then goes to the first free shooting stand. The time a skier spends at either of the shooting stands is independent of the other skiers and exponentially distributed with mean 30 seconds, i.e. with rate $\mu = 1/30$.

Let $X(t)$ denote the number of skiers in the stadium at time t, i.e. skiers who are either shooting or waiting in line until a shooting stand becomes free. We assume that $X(0) = 0$.

a)
- Explain briefly why $X(t)$ is a birth-death process and give all birth and death rates.
- If $X(t) = 3$, what is the expected time until all these three skiers have finished shooting.

b) Starting at time 0, what is the expected time until $X(t) = 3$ for the first time.

In the remaining questions, first express the answers as functions of λ and μ. Thereafter, compute the numerical answer for the parameter values given.

c)
- Derive the limiting probabilities for $X(t)$.
  (You can use that: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, if $|a| < 1$.)
- In the long run what proportion of skiers can start shooting immediately after arrival (i.e. without first waiting until a shooting stand becomes free)?

d)
- Compute the expected number of skiers in the stadium after a long time has passed.
- Use Little's formula to find the average amount of time each skier spends in the stadium.

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1, S_2, \ldots, S_n | N(t)}(s_1, s_2, \ldots, s_n | n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$