5.3 The Poisson process

A stochastic process is called counting process $N(t)$, $t \geq 0$, if $N(t)$ represents the total number of events by time t.

$N(t)$ must satisfy:

1. $N(t) \geq 0$
2. $N(t) \in \{0, 1, 2, \ldots\}$
3. If $s < t \Rightarrow N(s) \leq N(t)$.
4. For $s < t$ denotes $N(t) - N(s)$ the number of events in interval $(s, t]$.

Independent increments

Definition:

$N(t)$, $t \geq 0$, is said to possess independent increments if $N(t_2) - N(t_1)$ and $N(s_2) - N(s_1)$ are independent for $s_1 < s_2 < t_1 < t_2$. That means the numbers of events in disjoint intervals are independent.

Stationary increments/Time homogeneity

Definition:

$N(t)$, $t \geq 0$, is said to have stationary increments or to be time homogeneous, if the distribution of $N(t) - N(s)$ only depends on $t - s$. That means if the number of events in any interval $(s, s+t)$ has the same distribution for all s.

("Intervals of the same length behave statistically in the same fashion")

Definition of a Poisson process

The counting process $\{N(t), t \geq 0\}$ is called Poisson process with event (arrival) rate $\lambda > 0$ if:

i) $N(0) = 0$

ii) $\{N(t), t \geq 0\}$ has independent increments.

iii) Small interval probabilities. For VERY small h:

$$P(N(t+h) - N(t) = k) \approx \begin{cases} 1 - \lambda h & \text{if } k = 0 \\ \lambda h & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

Comments to iii)

- Step heights of the path have with probability 1 the height 1.
- no simultaneous events.

Number of events in interval of length t?

Goal: Get probability mass function for k arrivals in interval of length t. We know what happens in small intervals.

Idea: Split the interval of length t into many small intervals.

See blackboard

Number of events in interval of length t

The number of events in any interval of length t is a Poisson random variable with mean λt and variance λt, i.e
$N(t) \sim \text{Poisson}(\lambda t)$ or more generally
$N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$.

Hence, if you double the time you expect twice as many events. If you double λ, you expect twice as many events.

Interarrival and waiting time distribution

Let $\{N(t), t \geq 0\}$ be a Poisson process with arrival rate λ. Then we define:

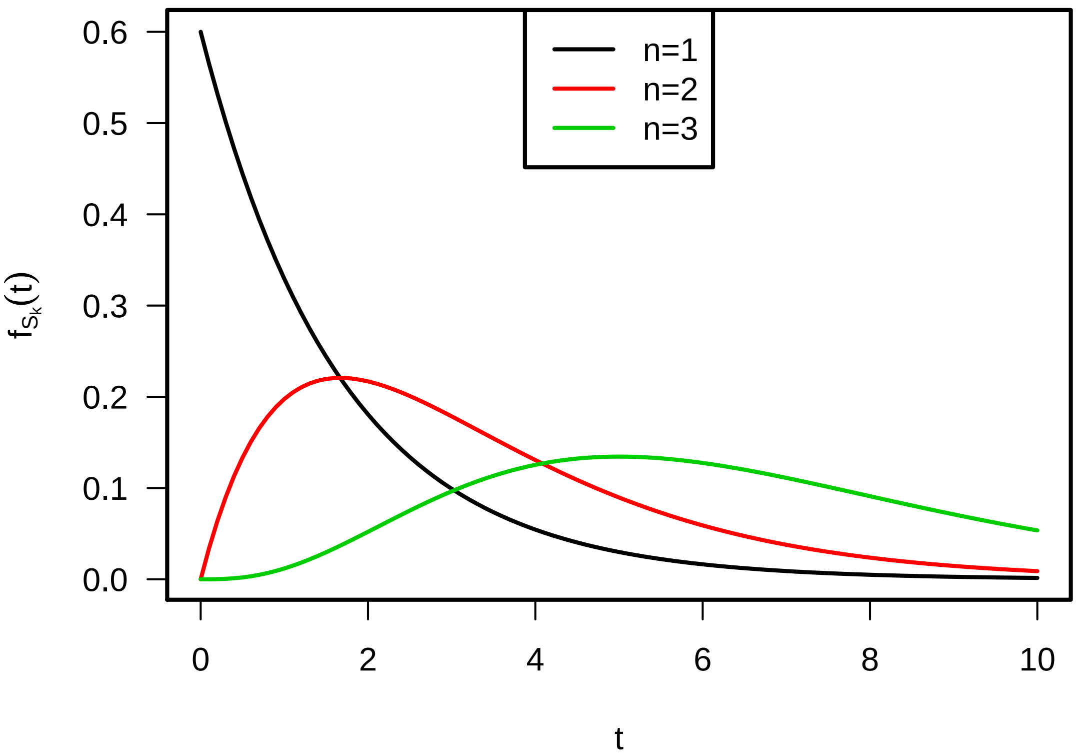
- T_n: time between event $n-1$ and event n (interarrival time)
- $S_n = \sum_{i=1}^{n} T_i$ waiting time until the n-th event.

Theorem

$T_n, n = 1, 2, \ldots$ are independent and identically exponential distributed, $T_n \sim Exp(\lambda)$.

Using convolution we get $S_n \sim Gamma(n, \lambda)$. For an alternative argument see blackboard.

Illustration of waiting time distribution for $\lambda = 0.6$



Example - Fishing

Assume that fish are caught according to a Poisson process with rate $\lambda = 0.6$ per hour.

- Fish for two hours.
- If no fish continue until first catch.

Example - Fishing II

a) P(fish for more than two hours)

b) P(fish more than two and less than five hours)

c) P(catch at least two fish)

d) E(future fishing time | fished for four hours)

e) E(total fishing time)