TMA4265 Stochastic Modelling: Solution exercise 1

Week 34

Problem 1: Coin toss

a) Let B denote the event that we flipped the biased coin, and B' the event that we flipped the normal coin. Further, let $H1$ denote the event that the first toss results in a head and $H2$ denote the event that the second toss results in a head.

The probability that we flipped the biased coin is

$$P(B|H1) = \frac{P(B \cap H1)}{P(H1)} = \frac{P(B)P(H1|B)}{P(B)P(H1|B) + P(B')P(H1|B')} = \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}$$

by Bayes' theorem and the law of total probability.

b) We flip the same coin one more time, and the second outcome is also a head. Thus, we are given that both the first and the second outcome is a head. The probability that we are flipping the biased coin is then given by

$$\begin{aligned} P(B|H1, H2) &= \frac{P(H1, H2|B)P(B)}{P(H1, H2)} \\ &= \frac{P(B)P(H1|B)P(H2|H1, B)}{P(B)P(H1|B)P(H2|H1, B) + P(B')P(H1|B')P(H2|H1, B')} \\ &= \frac{\frac{1}{2} \cdot 1 \cdot 1}{\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{4}{5}. \end{aligned}$$

Note that $P(H2|H1, B) = P(H2|B)$ and $P(H2|H1, B') = P(H2|B')$: The second outcome is independent of the first outcome given that we are throwing the same coin.

We then verify these results by a simulation.

We count the number of times the outcome is (H,H) and the *biased* coin was flipped...

```
biasedcount=0
```

...and the number of times the outcome is (H,H) and the *unbiased* coin was flipped.

```
unbiasedcount=0
```

Next, we run the following loop 100 times.

```
N=100
for(i in 1:N){
  cointype=runif(1) # We use runif to simulate which coin we throw.
  if(cointype<0.5){ # We throw the biased coin with 50% probability.
    biasedcount=biasedcount+1 # For the biased coin we get two heads in a row
    # for sure. Count.
  }else{ # We throw the unbiased coin with 50% probability.
    throw1=runif(1) # If this is lower than 0.5 -> First throw is H.
    throw2=runif(1) # If this is lower than 0.5 -> Second throw is H.

    if(throw1<0.5 & throw2<0.5){ # If both were under 0.5, we got two heads
      # in a row with the unbiased coin.
      unbiasedcount=unbiasedcount+1 #Count.
    }

  }
}
```

We can calculate the following fraction: Number of times two heads came from biased coin / (number of times two heads came from biased coin + number of times two heads came from unbiased coin). This should be around 4/5 (or 0.8) if `N` is large enough.

```
(biasedcount)/(biasedcount+unbiasedcount)
```

```
## [1] 0.8059701
```

Problem 2: Insurance claims

a) The moment generating function of the Normal distribution is given by $M_{Y_i}(t) = E(e^{Y_i t}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ where $Y_i \sim \mathcal{N}(\mu, \sigma^2)$. We are going to use this to find the $k'th$ moments of the log-Gaussian distribution:

$$E(C_i^k) = E(e^{Y_i k}) = e^{\mu k + \frac{\sigma^2 k^2}{2}}.$$

Here we use that $C_i = e^{Y_i}$ and recognize $E(e^{Y_i k})$ as the moment generating function of the Normal distribution. Setting $k = 1$ we obtain an expression for the expected value of the log-Gaussian distribution:

$$E(C_i) = E(C_i^1) = e^{\mu \cdot 1 + \frac{\sigma^2 \cdot 1^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}.$$

Now we can easily derive an expression for the variance of the log-Gaussian distribution as well:

$$Var(C_i) = E(C_i^2) - E(C_i)^2 = e^{\mu \cdot 2 + \frac{\sigma^2 \cdot 2^2}{2}} - (e^{\mu + \frac{\sigma^2}{2}})^2 = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1).$$

b) Double expectation gives

$$E(\sum_{i=1}^{N} C_i) = E(E(\sum_{i=1}^{N} C_i | N)).$$

Since

$$E(\sum_{i=1}^{N} C_i | N) = \sum_{i=1}^{N} E(C_i) = \sum_{i=1}^{N} \exp(\mu + \frac{\sigma^2}{2}) = N \exp(\mu + \frac{\sigma^2}{2}),$$

we get

$$E(\sum_{i=1}^{N} C_i) = E(E(\sum_{i=1}^{N} C_i|N)) = E(N\exp(\mu+\frac{\sigma^2}{2})) = E(N)\exp(\mu+\frac{\sigma^2}{2}) = \lambda\exp(\mu+\frac{\sigma^2}{2}),$$

Inserting $\mu = 2$, $\sigma = 1$ and $\lambda = 6$ into the above expression we get that the expected total sum of claim amounts is $E(\sum_{i=1}^{N} C_i)$=1.339 mill. kr.

Double variance gives

$$\begin{aligned}
Var(\sum_{i=1}^{N} C_i) &= E(Var(\sum_{i=1}^{N} C_i|N)) + Var(E(\sum_{i=1}^{N} C_i|N)) \\
&= E(\sum_{i=1}^{N} Var(C_i)) + Var(\sum_{i=1}^{N} E(C_i)) \\
&= E(N[(\exp(\sigma^2)-1)\exp(2\mu+\sigma^2)]) + Var(N\exp(\mu+\frac{\sigma^2}{2})) \\
&= [(\exp(\sigma^2)-1)\exp(2\mu+\sigma^2)]E(N) + [\exp(\mu+\frac{\sigma^2}{2})]^2 Var(N) \\
&= (\exp(\sigma^2)-1)\exp(2\mu+\sigma^2)\lambda + [\exp(\mu+\frac{\sigma^2}{2})]^2\lambda.
\end{aligned}$$

Inserting $\mu = 2$, $\sigma = 1$ and $\lambda = 6$ into the above expression we get that the variance of the total sum of claim amounts is $Var(\sum_{i=1}^{N} C_i|N) = 0.812$ mill. kr.

c) We simulate the situation on the problem above, using R.

```
mu=-2
sig=1
lambda=6
B=1000 # Number of years simulated.

Ctot=c()
for(i in 1:B){
  N=rpois(1,lambda) # Draw number of insurance claims in years i.

  Ctot[i]=sum(rlnorm(N,meanlog=mu,sdlog=sig)) # Draw N insurance claims from
  # lognormal distribution and sum them.

}

mean(Ctot)
```

```
## [1] 1.336717
```

```
var(Ctot)
```

```
## [1] 0.7895258
```