TMA4265 Stochastic Modelling: Solution exercise 2

Week 35

Problem 1

Given the markov chain $\{X_n : n = 0, 1, 2, \ldots\}$ with the state space $\Omega = \{A, B, C\}$, we derive the following transition matrix structures:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.6 & 0.1 \end{pmatrix} \end{array}$$

$$P^2 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.42 & 0.26 & 0.32 \\ 0.22 & 0.6 & 0.18 \\ 0.36 & 0.33 & 0.31 \end{pmatrix} \end{array}$$

$$P^3 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.268 & 0.512 & 0.220 \\ 0.376 & 0322 & 0.302 \\ 0.294 & 0.471 & 0.235 \end{pmatrix} \end{array}$$

Probability distribution at initial state X_0

$P(X_0 = A) = 0.2, P(X_0 = B) = 0.5, P(X_0 = C) = 0.3.$

a)

$$\begin{aligned} P(X_3 = A) &= \sum_{k \in \{A,B,C\}} P(X_3 = A | X_0 = k) P(X_0 = k) \\ &= P(X_3 = A | X_0 = A) P(X_0 = A) + P(X_3 = A | X_0 = B) P(X_0 = B) \\ &+ P(X_3 = A | X_0 = C) P(X_0 = C) \\ &= 0.268 \cdot 0.2 + 0.367 \cdot 0.5 + 0.294 \cdot 0.3 \\ &= 0.3298. \end{aligned}$$

$$
\begin{aligned}
P(X_3 = B) &= \sum_{k\in\{A,B,C\}} P(X_3 = B|X_0 = k)P(X_0 = k) \\
&= P(X_3 = B|X_0 = A)P(X_0 = A) + P(X_3 = B|X_0 = B)P(X_0 = B) \\
&+ P(X_3 = B|X_0 = C)P(X_0 = C) \\
&= 0.512 \cdot 0.2 + 0.322 \cdot 0.5 + 0.471 \cdot 0.3 \\
&= 0.4047.
\end{aligned}
$$

$$
\begin{aligned}
P(X_3 = C) &= 1 - P(X_3 = A) - P(X_3 = B) \\
&= 1 - 0.3298 - 0.4047 \\
&= 0.2655.
\end{aligned}
$$

b)

Since X_n is a markov chain,

$$
\begin{aligned}
P(X_3 = A|X_1 = B, X_0 = A) &= P(X_3 = A|X_1 = B) \\
&= P(X_2 = A|X_0 = B) \quad (\text{Check } P^2_{BA}) \\
&= 0.22.
\end{aligned}
$$

c)

Since X_n is a markov chain,

$$
\begin{aligned}
P(X_3 = A|X_2 = C, X_1 = B, X_0 = A) &= P(X_3 = A|X_2 = C) \\
&= P(X_1 = A|X_0 = C) \\
&= 0.3.
\end{aligned}
$$

d)

$$
\begin{aligned}
P(X_6 = A|X_3 = C) &= P(X_3 = A|X_0 = C) \quad (\text{Check } P^3_{CA}) \\
&= 0.294.
\end{aligned}
$$

e)

$$
\begin{aligned}
P(X_3 = C|X_6 = A) &= \frac{P(X_3 = C, X_6 = A)}{P(X_6 = A)} \\
&= \frac{P(X_6 = A|X_3 = C)P(X_3 = C)}{\sum_{k\in\{A,B,C\}} P(X_6 = A|X_3 = k)P(X_3 = k)} \\
&= \frac{P(X_3 = A|X_0 = C)P(X_3 = C)}{\sum_{k\in\{A,B,C\}} P(X_3 = A|X_0 = k)P(X_3 = k)} \\
&= \frac{0.294 \cdot 0.2655}{0.294 \cdot 0.2655 + 0.376 \cdot 0.4047 + 0.268 \cdot 0.3298} \\
&= 0.245.
\end{aligned}
$$

Problem 2

a)

Suppose we model the daily weather as a Markov Chain $\{X_n : n = 0, 1, 2, \ldots\}$, where n denotes the day. We then have two states in our state space Ω

- State 0 : Rain
- State 1 : No rain

Let us define $P_{ij} = P(X_1 = j | X_0 = i)$ be the probability that an state j occurs tomorrow given state i occurs today. Then, $P_{01} = 0.7$ and $P_{10} = 0.25$.

Our transition probabilities for this Markov chain should satisfy the conditions:

- $P_{ij} > 0$
- $\sum_{j=1,2} P_{ij}$ for all $i \in \Omega$.

Hence our transition probabilities is given by:

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

b)

Let n_0 be on Monday. Then $X_0 = 0$. We want to obtain $P(X_4 = 0 | X_0 = 0) = P^4_{00}$.

$$P^4 = P \cdot P \cdot P \cdot P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.4769 & 0.5231 \\ 0.4359 & 0.5641 \end{pmatrix} \end{array}$$

Therefore the probability that it will rain on Friday given that today is Monday and it is raining is 0.48.

c)

The long term transition probability for rain can be obtained analytically from $\lim_{n\to\infty} P^n$. In R , we compute P^{100} and get:

$$P^{100} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.4545 & 0.5454 \\ 0.4545 & 0.5454 \end{pmatrix} \end{array}$$

It can be observed that in the long-run, the initial state does not affect whether there will be “rain” or “no rain” today.

d)

See Blackboard for code.

e)

See Blackboard for code.

Problem 3

Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a Markov Chain with the given transition probabilities and let $T = min\{n \geq 0 | X_n = 0 \quad \text{or} \quad X_n = 3\}$ denote the time of absorption.

We assume that the starting state is state 1.

a)

We are interested in $E(T|X_0 = 1)$ and we use the first step analysis to find this quantity.

Let $v_i = E(T = |X_0 = i)$. Then $v_0 = v_3 = 0$

Further,

$$\begin{aligned} v_1 &= 1 + \sum_{j=0}^{3} v_j P_{1j} \\ &= 1 + v_1 P_{11} + v_2 P_{12} \\ &= 1 + 0.4v_1 + 0.1v_2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} v_2 &= 1 + \sum_{j=0}^{3} v_j P_{2j} \\ &= 1 + v_1 P_{21} + v_2 P_{22} \\ &= 1 + 0.1v_1 + 0.6v_2 \end{aligned} \tag{2}$$

Solving the linear system of the two equations, we get $v_1 \approx 2.17$. Hence,the expected time to absorption is 2.17.

b)

We are interested in $P(X_T = 0|X_0 = 1)$ and we use the first step analysis to find this quantity.

Let $u_i = P(X_T = 0|X_0 = i)$. Then $u_0 = 1$ and $u_3 = 0$.

Further,

$$\begin{aligned} u_1 &= \sum_{j=0}^{3} u_j P_{1j} \\ &= u_0 P_{10} + u_1 P_{11} + u_2 P_{12} \\ &= 0.1 + 0.4u_1 + 0.1u_2 \end{aligned} \tag{3}$$

and

$$\begin{aligned} u_2 &= \sum_{j=0}^{3} u_j P_{2j} \\ &= u_0 P_{20} + u_1 P_{21} + u_2 P_{22} \\ &= 0.2 + 0.1u_1 + 0.6u_2 \end{aligned} \tag{4}$$

Solving the linear system of the two equations, we get $u_1 \approx 0.261$. Hence, the probability that the Markov chain ends up in state 0 is 0.261.