TMA4265 Stochastic Modelling: Solution Exercise 3

Week 36

Problem 1

A Markov chain with transition probability matrix $\boldsymbol{P}$ is said to be regular if there exists some positive constant k such that all the elements of $\boldsymbol{P}^k$ are strictly positive. Using the notation of Pinsky and Karlin, we denote any nonzero element of $\boldsymbol{P}$ using a $+$. This gives the following solutions:

a)

$$\boldsymbol{P} = \begin{pmatrix} + & + \\ + & 0 \end{pmatrix} \implies \boldsymbol{P}^2 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}.$$

All the elements of $\boldsymbol{P}^2$ are nonzero, so the corresponding Markov chain is regular.

b) We recognise the transition probability matrix as the identity matrix. Thus, we have $\boldsymbol{P}^2 = \boldsymbol{P}$, and for all $k = 1, 2, \ldots$ we have $\boldsymbol{P}^k = \boldsymbol{P}$. Since not all the elements of $\boldsymbol{P}$ are nonzero, we can never find a k such that all the elements of $\boldsymbol{P}^k$ are nonzero. The corresponding Markov chain is therefore not regular.

c)

$$\boldsymbol{P} = \begin{pmatrix} + & + \\ 0 & + \end{pmatrix} \implies \boldsymbol{P}^2 = \begin{pmatrix} + & + \\ 0 & + \end{pmatrix}.$$

There is no change in which elements are nonzero after we multiply $\boldsymbol{P}$ with itself. Thus, for any $k = 1, 2, \ldots$, we find that the first element of the second row of $\boldsymbol{P}^k$ is nonzero. The corresponding Markov chain is therefore not regular.

d)

$$\boldsymbol{P} = \begin{pmatrix} + & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{pmatrix} \implies \boldsymbol{P}^2 = \begin{pmatrix} + & + & + \\ + & 0 & 0 \\ + & + & 0 \end{pmatrix} \implies \boldsymbol{P}^4 = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$

All the elements of $\boldsymbol{P}^4$ are nonzero, so the corresponding Markov chain is regular.

e) Looking at $\boldsymbol{P}$, we notice that, once you are in state 2 of the Markov chain, you have reached a capturing class. From state 2 one will always move to state 3 and from state 3 one will always move to state 2. Thus, we will never reach back to state 1. This means that the probability of e.g.
moving from state 3 to state 1 in k steps is zero for all $k = 1, 2, \ldots$, meaning that $P_{31}^{(k)} = 0$. Consequently, the corresponding Markov chain is not regular.

Problem 2

a) It is clear that the transition probability matrix $\boldsymbol{P}$ is regular. We therefore know that the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$ is the unique nonnegative solution of the equations

$$\pi_j = \sum_{i=0}^{2} \pi_i P_{ij}, \quad j = 0, 1, 2,$$

subject to the constraint that the sum of $\boldsymbol{\pi}$ is 1. The first 3 equations can be written in matrix form as

$$\boldsymbol{P}^T\boldsymbol{\pi} = \boldsymbol{\pi} \iff (\boldsymbol{P}^T - \boldsymbol{I})\boldsymbol{\pi} = 0,$$

while the constraint can be written as

$$\mathbf{1}^T\pi = 1,$$

where $\mathbf{1}$ is a 3-dimensional vector containing only ones. We therefore get the matrix equation

$$\begin{pmatrix} P_{11}-1 & P_{21} & P_{31} \\ P_{12} & P_{22}-1 & P_{32} \\ P_{13} & P_{23} & P_{33}-1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We use Gauss elimination to solve this system of equations:

$$\begin{pmatrix} -.6 & .4 & .2 & 0 \\ .45 & -.6 & .4 & 0 \\ .15 & .2 & -.6 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2/3 & -1/3 & 0 \\ 0 & -.3 & .55 & 0 \\ 0 & .3 & -.55 & 0 \\ 0 & 5/3 & 4/3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -14/9 & 0 \\ 0 & 1 & -11/6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 79/18 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 28/79 \\ 0 & 1 & 0 & 33/79 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 18/79 \end{pmatrix}.$$

This means that it is sunny 28/79 of the time, light rain 33/79 of the time, and heavy rain 18/79 of the time.

b) Denote the cost of getting to work C. If it rains, the cost is 80 kr, $P(C = 80) = \pi_1 + \pi_2$. If it does not rain, the cost is 0 kr, $P(C = 0) = \pi_0$.

In the long run, the average cost of getting to work is

$$E[C] = 0 \cdot \pi_0 + 80 \cdot (\pi_1 + \pi_2) = 80 \cdot \frac{51}{79} \approx 51.65.$$

c) On any given day, John either travels to work with a bus, or he doesn't. Thus, the random variable B_i which is 1 if he takes bus and 0 if he walks on day i of the month, is Bernoulli distributed with success probability $p = \pi_1 + \pi_2 = 51/79$.

The number of days he takes the bus during a 30 day period is therefore $N = \sum_{i=1}^{30} B_i$. The expected number of days he will take the bus during a 30 day period is then

$$E[N] = E[\sum_{i=1}^{30} B_i] = \sum_{i=1}^{30} E[B_i] = \sum_{i=1}^{30} p = 30p \approx 19.37.$$

This is larger than 18, so it seems wise of him to start buying 30 day tickets.

d)

```
# Create the probability transition matrix
P = matrix(c(.4, .4, .2, .45, .4, .4, .15, .2, .4), nrow = 3, ncol = 3)

N = 100000 # Number of days we are going to simulate

weather = rep(NA, N + 1) # The weather vector, which can take the values 0, 1 or 2
weather[1] = 0 # First day is sunny

set.seed(1)
for (i in 1:N) {
  previous_day = weather[i]
  weather_today_probabilities = P[previous_day + 1, ]
  # Sample from the vector [0, 1, 2] with probabilities weather_today_probabilities
  weather[i + 1] = sample(c(0, 1, 2), size = 1, prob = weather_today_probabilities)
```

```
}

# Test if the empirical probabilities are close to the theoretical probabilities
table(weather) / N # Empirical probabilities
```

```
## weather
##       0       1       2
## 0.35443 0.41846 0.22712
```

```
c(28 , 33, 18) / 79 # Theoretical probabilities
```

```
## [1] 0.3544304 0.4177215 0.2278481
```

```
# Count the number of days John takes the bus to work, and divide it by the number of 30 day periods
sum(table(weather)[c(2, 3)]) / (N / 30) # This is more than 18
```

```
## [1] 19.3674
```

Problem 3

Similarly to problem 2 a), we set up a matrix equation that combines

$$\boldsymbol{P}^T\boldsymbol{\pi} = \boldsymbol{\pi} \iff (\boldsymbol{P}^T - \boldsymbol{I})\boldsymbol{\pi} = 0$$

with the constraint

$$\mathbf{1}^T\pi = 1$$

and then we use Gauss-elimination to solve this. We get the matrix equation,

$$\begin{pmatrix} p-1 & 0 & q \\ q & p-1 & 0 \\ 0 & q & p-1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using the fact that $p + q = 1 \iff p - 1 = -q$, this is reduced to

$$\begin{pmatrix} -q & 0 & q \\ q & -q & 0 \\ 0 & q & -q \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

so we see that the limiting probabilities are independent of the values of p and q. Gauss elimination gives

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the limiting probabilities are $\pi_0 = \pi_1 = \pi_2 = 1/3$.