TMA4265 Stochastic Modelling: Solution Exercise 4

Week 37

Problem 1

Given the markov chain $\{X_n : n = 0, 1, 2, \ldots\}$ and the transition matrix $\mathbf{P}$.

a) The chain is reducible as state 0 is not accessible from state 4.

We observe that $0 \sim 1$, $2 \sim 3$ and $4 \sim 5$, and there can be at most the three equivalence classes with these pairs. Further, $0 \not\sim 2$, $0 \not\sim 4$ and $2 \not\sim 4$, and we find:

Equivalence classes: $ {0,1}$, $ {2,3}$, $ {4,5}$.

b) Since the transition probability matrix diagonal has strictly positive values, $P_{ii} > 0$, all the states have period 1, i.e. we have an aperiodic Markov Chain.

c)

- $\{0, 1\}$: It is not possible to leave the equivalence class and it is of finite size. Therefore, we are expected to visit each state infinitely many times and the equivalence class is **recurrent**.
- $\{2, 3\}$: It is possible to leave the equivalence class. Therefore, there is a strictly positive probability of never returning for each state, and the equivalence class is **transient**.
- $\{4, 5\}$: It is not possible to leave the equivalence class and it is of finite size. Therefore, we are expected to visit each state infinitely many times and the equivalence class is **recurrent**.
- A state is **absorbing** if once the process has entered the state, it is impossible to leave. There are no such states.

Problem 2

Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a Markov chain with finite state space $\{0, 1, 2, \ldots, N\}$ and probability transition matrix $\mathbf{P}$. Assume that the Markov chain is aperiodic and irreducible.

Since all states communicate, there is only one equivalence class. Since there is only one equivalence class and it has finite size, all states are expected to be visited infinitely many times and the Markov chain is **recurrent**.

Proving regularity requires combining the properties of irreducibility and aperiodicity:

1. Since the Markov chain is irreducible, all states communicate and for each pair of states i and j, there exists a positive integer n_{ij} such that $P_{ij}^{(n_{ij})} > 0$.
2. Since the Markov chain is aperiodic, there exists for each state j, a positive integer m_j such that $P_{jj}^{(m)} > 0$ for all $m \geq m_j$.
3. Combining 1. and 2. leads to the conclusion that $P_{ij}^{(n_{ij}+m)} \geq P_{ij}^{(n_{ij})} P_{jj}^{(m)} > 0$ for all $m \geq m_j$.

Summarizing, for any pair of states i and j, there exists a positive integer $m_{ij} = n_{ij} + m_j$ such that $P_{ij}^{(n)} > 0$ for all $n \geq m_{ij}$. By taking $k = \max_{ij}\{m_{ij}\}$, we have that $P_{ij}^{(k)} > 0$ for all states i and j, and the Markov chain is regular.

Problem 3

a) The chain is irreducible since all the three states communicate to one another.

b) Let $\pi = \{\pi_1, \pi_2, \pi_3\}$ be the limiting probability.

Then

$$\pi^T = \pi^T P$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.05 & 0.8 & 0.15 \\ 0 & 0.1 & 0.9 \end{pmatrix}$$

subject to

$$\pi_1 + \pi_2 + \pi_3 = 1$$

This implies that:

$$\begin{aligned} & \pi_1 = 0.9\pi_1 + 0.05\pi_2 \\ \implies & \pi_1 = 0.5\pi_2 \end{aligned}$$

$$\begin{aligned} & \pi_3 = 0.15\pi_2 + 0.9\pi_3 \\ \implies & \pi_3 = 1.5\pi_2 \end{aligned}$$

Solving the equations gives us $\pi_1 = 1/6$, $\pi_2 = 1/3$ and $\pi_3 = 1/2$.

c)

```
P=matrix(c(0.9,0.1,0,
           0.05,0.8, 0.15,
           0, 0.1,0.9),nrow=3,byrow=TRUE)

i <- 1
P_old <- array(data = NA, dim = c(3,3,301))
P_old[,,1] <- P
while(i < 301){
  P_new <- P_old[,,i]%*% P
  P_old[,,i+1] <- P_new
  i <- i+1
}
print(P_300 <- P_old[,,300])
```

```
##           [,1]      [,2] [,3]
## [1,] 0.1666667 0.3333333  0.5
## [2,] 0.1666667 0.3333333  0.5
## [3,] 0.1666667 0.3333333  0.5
```

Problem 4

a) Each throw is independent of other throws. Only the previous state (number of heads in a row) affects the probability of transitioning into a new state. The transition matrix is:

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & \dots \\ 0.5 & 0 & 0.5 & 0 & 0 & \dots \\ 0.5 & 0 & 0 & 0.5 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The state space is $\Omega = \mathbb{N}_0$.

b)The Markov Chain is irreducible. Let i, j be any two states. We can always go from state i to 0, from there 0 to 1 and so on, until we have reached state j.

c) By the definition, the period of state i is

$$\gcd\{n > 0 : P_{ii}^{(n)} > 0\}.$$

Since all states communicate with each other, we can choose one state and find the period of that state since all other states have the same period. For $i = 0$,

$$P_{0,0} > 0,$$

and we can return in any number of time steps and the period is $\gcd\{1, 2, 3, 4, \dots\} = 1$. This means that all states have period 1.

d) Since all the states communicate with each other, we can choose one state and decide if it is recurrent or transient. Note that since the equivalence class is infinite, the fact that we cannot leave the equivalence class is not enough to decide that it is recurrent.

Recurrent can be seen directly from $f_{00}^{(n)} = (1/2)^n$ for $n > 0$. Which implies that $f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = \sum_{n=1}^{\infty} 1/2^n = 1$.

e) States are actually positive recurrent. Consider state 0 and let

$$M = \text{“Time to first return to state 0”}.$$

Each throw is independent, with the same probability $p = 0.5$ of returning to state 0 (by throwing tails). Thus M must follow a geometric distribution and $m_0 = \mathrm{E}[M] = 1/(1 - 1/2) = 2$. Since $m_0 < \infty$, state 0 is positive recurrent, and it follows that all states are positive recurrent.

Positive recurrent in combination with aperiodicity and irreducibility means it has a unique limiting distribution; see Theorem 4.4. Using the fact that the mean duration between visits to state 0 is 2, we must have that

$$\pi_0 = 1/m_0 = 1/2.$$

Then for states 1 and 2, we find

$$\pi_1 = \sum_{k=0}^{\infty} \pi_k P_{k1} = \pi_0 P_{01} = \frac{1}{4},$$
$$\pi_2 = \sum_{k=0}^{\infty} \pi_k P_{k2} = \frac{1}{2}\pi_1 = \frac{1}{8}.$$

We want the proportion of time where the sequence ends in three or more heads, i.e., we are in states $3, 4, \ldots$,

$$\sum_{i=3}^{\infty} \pi_i = 1 - \pi_0 - \pi_1 - \pi_3 = \frac{1}{8}.$$