TMA4265 Stochastic Modelling: Solution Exercise 5

Week 38

Problem 1

a)

(i): The probability that there are no defects in the first foot of the filament is

$$P(X(1) = 0) = e^{-\lambda} = e^{-2}.$$

(ii): Since the process has independent increments, the conditioning does not affect the answer:

$$P(X(2) = 0|X(1) = 0) = P(X(1) = 0) = e^{-\lambda} = e^{-2}.$$

b)

$$p_0 = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}.$$

$$(p_k/p_{k-1}) = \frac{e^{-\lambda}\lambda^k(k-1)!}{e^{-\lambda}\lambda^{k-1}k!} = \frac{\lambda}{k}$$

c) By using Theorem 5.1 from the book, the definition of conditional probability and independence between X, Y, we get

$$\begin{aligned} P(X = k|N = n) &= \frac{P(X = k \cap Y = n - k)}{P(X + Y = n)} \\ &= \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} \\ &= \frac{\frac{\alpha^k}{k!}e^{-\alpha}\frac{\beta^{n-k}}{(n-k)!}e^{-\beta}}{\frac{(\alpha+\beta)^n}{n!}e^{-(\alpha+\beta)}} \\ &= \binom{n}{k}p^k(1-p)^{n-k}, \end{aligned}$$

where $p = \frac{\alpha}{\alpha+\beta}$. Thus, the conditional distribution of X given $X + Y = n$ is binomial with parameters $p = \alpha/(\alpha + \beta)$ and n.

d) The probability that the first particle appears after 3 min is the probability that no particle appears during the first 3 min:

$$P(X(3) = 0) = e^{-3\lambda} = e^{-6}.$$

e) (i): The waiting time of a Poisson process with rate λ is exponentially distributed with parameter λ.

$$P(3 < W_1 < 5) = P(W_1 < 5) - P(W_1 \leq 3) = e^{-6} - e^{-10}$$

(ii): This is the probability that $X(5) = k + 1$ given that $X(3) = k$, for any $k = 0, 1, \ldots,$:

$$P(X(5) = k + 1 | X(3) = k) = P(X(2) = 1) = \frac{2\lambda}{1} e^{-2\lambda} = 4e^{-4}$$

f) If $\lambda = 6$ is the hourly rate, then $1/10$ is the rate per minute. Using the conditional probability and the independence of increments, we get

$$\begin{aligned} P(X(15) = 1 | X(60) = 1) &= \frac{P(X(15) = 1 \cap X(60) = 1)}{P(X(60) = 1)} \\ &= \frac{P(X(15) = 1) P(X(60) - X(15) = 0)}{P(X(60) = 1)} \\ &= \frac{15\lambda e^{-15\lambda} \cdot e^{-45\lambda}}{60\lambda e^{-60\lambda}} \\ &= \frac{1}{4}. \end{aligned}$$

g) We start by examining the distribution of the first arrival time,

$$P(W_1 \leq t | X(1) = n) = 1 - P(W_1 > t | X(1) = n) = 1 - P(X(t) = 0 | X(1) = n).$$

It is clear that for all $t \geq 1$, this probability is equal to 0, since $X(1) = n$ and $n > 0$. We use Bayes rule to simplify the expression:

$$P(X(t) = 0 | X(1) = n) = \frac{P(X(1) = n | X(t) = 0) P(X(t) = 0)}{P(X(1) = n)} = \frac{P(X(1 - t) = n) P(X(t) = 0)}{P(X(1) = n)}, \ 0 \leq t < 1.$$

All these probabilities are known,

$$\frac{P(X(1 - t) = n) P(X(t) = 0)}{P(X(1) = n)} = \frac{\frac{(\lambda(1-t))^n}{n!} e^{-\lambda(1-t)} \cdot e^{-\lambda t}}{\frac{\lambda^n}{n!} e^{-\lambda}} = (1 - t)^n.$$

Thus,

$$P(W_1 \leq t | X(1) = n) = 1 - (1 - t)^n, \ 0 \leq t < 1,$$

and the probability density function of $[W_1 | X(1) = n]$ is

$$f_{W_1 | X(1)}(w | n) = n(1 - w)^{n-1}, \ 0 \leq w < 1$$

(This might be recognised as the probability density function of the minimum of n uniformly distributed random variables between 0 and 1.)

The mean of $[W_1 | X(1) = n]$ is

$$\int_0^1 w \cdot n(1 - w)^{n-1} \mathrm{d}w = \frac{1}{n + 1},$$

where the integral is solved using integration by parts.

Problem 2: Football

Let $R(t)$ be the number of goals scored by Rosenborg, let $V(t)$ be the number of goals scored by Vålerenga and let $N(t)$ be the total number of goals scored during a match where $0 \leq t \leq 1$ is the proportion of a match that has been played.

a) The total number of goals scored during one match is Poisson distributed with average $\lambda_{tot} = \lambda_r + \lambda_v$:

$$P(N(1) = n) = \frac{(\lambda_r + \lambda_v)^n}{n!} e^{-(\lambda_r + \lambda_v)},$$

because the distribution of a sum of Poisson distributions is Poisson distributed.

b)

$$P(N(0.5) = 0) = e^{-0.5(\lambda_r + \lambda_v)} \approx 0.2019$$

c) The probability that the final result is 2-2 is given by

$$P(R(1) = 2, V(1) = 2) = P(R(1) = 2)P(V(1) = 2) = \frac{\lambda_r^2}{2} e^{-\lambda_r} \frac{\lambda_v^2}{2} e^{-\lambda_v} \approx 0.0587,$$

where we use that the number of goals scored by Rosenberg is independent of the number of goals scored by Vålerenga.

d) The time until the first goal in the match is exponentially distributed with expected value $\lambda_{tot}^{-1} = (\lambda_r + \lambda_v)^{-1} = 1/3.2$. Since we measure time in "proportion of 90 min'', this means that the expected time until first goal is $90/3.2\text{min} \approx 28.125\text{min}$.

e) Because of the memoryless property, the number of goals scored from $t = 15\text{min}$ to $t = 45\text{min}$ is independent of the number of goals scored from $t = 0\text{min}$ to $t = 15\text{min}$. 45 minutes is half the match, while 15 minutes is one sixth of the match. Thus

$$\begin{aligned} P(V(1/2) > 1 | V(1/6) = 0) &= P(V(1/2) > 1 | V(1/6) = 0) \\ &= P(V(1/3) > 1) \\ &= 1 - P(V(1/3) = 0) \\ &= 1 - e^{-\lambda_v/3} \\ &\approx 0.3297. \end{aligned}$$

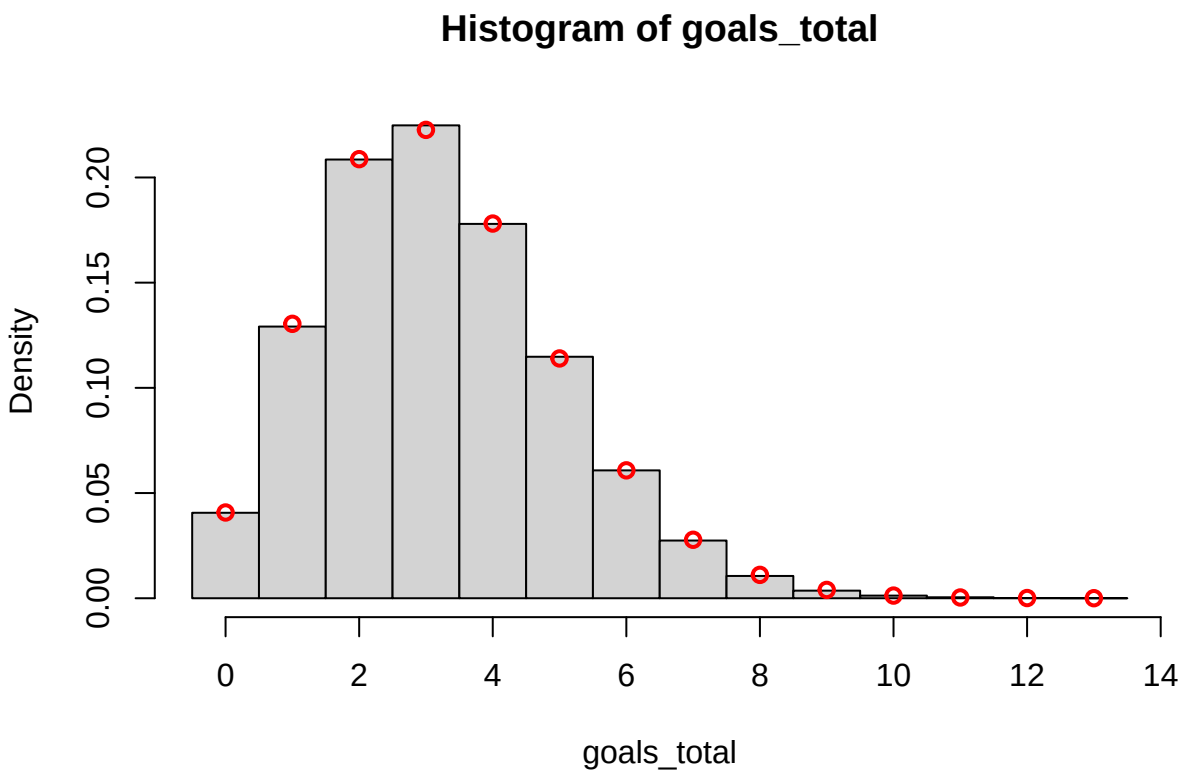
f)

```
N = 100000 # Number of matches we simulate

# Mean number of goals per match.
lambda_vif = 1.2
lambda_rbk = 2
lambda_total = lambda_vif + lambda_rbk
```

```
# ===========
# a)
# ===========
# Sample the number of goals in all the N matches
set.seed(1)
goals_vif = rpois(N, lambda = 1.2)
goals_rbk = rpois(N, lambda = 2)
goals_total = goals_vif + goals_rbk

# Plot the total number of goals, and compare with the@
# theoretical distribution: Poisson with lambda = 3.2
hist(goals_total, freq = FALSE, breaks = 0:max(goals_total + 1) - .5)
points(0:max(goals_total), dpois(0:max(goals_total), lambda = lambda_total)
       , col = "red", lwd = 2)
```




```
# ===========
# b)
# ===========
set.seed(1)
lambda_firsthalf = lambda_total / 2
goals_firsthalf = rpois(N, lambda = lambda_firsthalf)
prob_zero_goals_firsthalf = mean(goals_firsthalf == 0)

# Compare the estimated probability of zero goals with the theoretical probability
prob_zero_goals_firsthalf
```

```
## [1] 0.20354
```

```
exp(-lambda_firsthalf)
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```
## [1] 0.2018965
```

```
# ===========
# c)
# ===========
# Probability that the result is 2-2.
prob_22 = mean(goals_vif == 2 & goals_rbk == 2)

# Compare the estimated and theoretical probabilities
prob_22
```

```
## [1] 0.05878
```

```
.5 * lambda_vif^2 * exp(-lambda_vif) * .5 * lambda_rbk^2 * exp(-lambda_rbk)
```

```
## [1] 0.05869757
```

```r
# ===========
# d)
# ===========
set.seed(1)
# Sample the time of the first goal, and transform so the values have unit minutes
firstgoal = rexp(N, rate = lambda_total) * 90
mean_time_to_first_goal = mean(firstgoal)

# Compare the estimated and theoretical mean times
mean_time_to_first_goal
```

```
## [1] 28.20087
```

```r
90 / lambda_total
```

```
## [1] 28.125
```

```r
# ===========
# e)
# ===========
set.seed(1)
# Sample the time of the first goal, and transform so the values have unit minutes
firstgoal_vif = rexp(N, rate = lambda_vif) * 90
firstgoal_rbk = rexp(N, rate = lambda_rbk) * 90
# The first goal is the minimum time of the first goal for the two teams
firstgoal = pmin(firstgoal_vif, firstgoal_rbk)

no_goal_first_quarter = firstgoal > 15
vif_first_goal_before_halftime = firstgoal_vif < 45
conditional_prob = mean(no_goal_first_quarter & vif_first_goal_before_halftime)/
  mean(no_goal_first_quarter)

# Compare the estimated and theoretical probabilities
conditional_prob
```

```
## [1] 0.3274521
```

```r
1 - exp(-lambda_vif / 3)
```

```
## [1] 0.32968
```