TMA4265 Stochastic Modelling: Solution Exercise 6

Week 41

Problem 1: Exercise 6.1.1. and 6.1.2. in the book

a) From equation (6.5),

$$P_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{-\lambda_n x} P_{n-1}(x) dx, \quad n = 0, 1, 2, \ldots$$

Probabilities through integration:

$$P_0(t) = e^{-\lambda_0 t},$$

$$P_1(t) = \lambda_0 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 x} P_0(x) dx = \frac{1}{2} e^{-3t}(e^{2t} - 1),$$

$$P_2(t) = \lambda_1 e^{-\lambda_2 t} \int_0^t e^{\lambda_2 x} P_1(x) dx = \frac{3}{2}(e^{-t} + e^{-3t} - 2e^{-2t})$$

The same approach is done for $P_3(t)$.

b) Let S_0, S_1 and S_2 be the sojourn times. We want to estimate $W_3 = S_0 + S_1 + S_2$ and we know that $S_0 \sim \text{Exp}(\lambda_0)$, $S_1 \sim \text{Exp}(\lambda_1)$, $S_2 \sim \text{Exp}(\lambda_2)$. Since the sojourn times are mutually independent (p. 280),

$$EW_3 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{11}{6}$$

c) $E(W_1 + W_2 + W_3) = E(W_1) + E(W_2) + E(W_3) = 3\frac{1}{\lambda_0} + 2\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$

d) $\text{Var}(W_3) = \frac{1}{\lambda_0^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$.

Problem 2: Parking garage

a) Let us define the rate of the arrival of cars to be $\lambda = 0.5$cars/min. Let \$N(t)= \$ the number of customers in time $(0, t)$. This is Poisson distributed with parameter λt.

- We calculate the probability that no car has arrived at $08:05$.

$$P(N(5) = 0) = \exp(-\lambda \cdot 5) = \exp(-0.5 \cdot 5) = 0.082$$

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- We calculate the expected number of cars that will arrive during the first 15 minutes of the opening hours.

$$E(N(15)) = \lambda \cdot 15 = 0.5 \cdot 15 = 7.5$$

- We calculate the probability that no cars arrive during the first 5 minutes given that two cars will arrive during the first 10 minutes.

$$\begin{aligned} P(N(5)=0|N(10)=2) &= \frac{P(N(5)=0, N(10)-N(5)=2)}{P(N(10)=2)} \\ &= \frac{\exp(-5\cdot\lambda)\frac{(\lambda\cdot 5)^2}{2}\exp(-5\cdot\lambda)}{\frac{(\lambda\cdot 10)^2}{2}\exp(-10\cdot\lambda)} \\ &= 0.25. \end{aligned}$$

b) Let C_i be the amount spent by customer i and X be the total amount paid by all $N(t)$ customers. That is:

$$X = C_1 + C_2 + \ldots + C_{N(t)}.$$

For the entire day (8 hours), the time in minutes is $t = 60 \cdot 8 = 480$ minutes. Using the double expectation:

$$E(X) = E(E(X|N(t)=n)) = E(C)E(N(t)) = 100 \cdot \lambda \cdot t = 100 \cdot 480 = 48000kr.$$

Using the double variance:

$$\begin{aligned} \mathrm{Var}(X) &= E(\mathrm{Var}(X|N(t)=n)) + \mathrm{Var}(E(X|N(t)=n)) \\ &= \mathrm{Var}(C)E(N(t)) + E(C)^2\mathrm{Var}(N(t)) \\ &= 10^2 \cdot \lambda 480 + 100^2 \cdot \lambda 480. \\ &= 2424000 \end{aligned}$$

c)

- We define the birth and death process arrivals as a Poisson process with intensity

$$\lambda_i = \lambda.$$

The departures are also run by the Poisson process. There are i customers that can leave and the minimum of i independent exponential distributions with rate μ is an exponential distribution with rate $i\mu$ and so the death rate is

$$\mu_i = i\mu.$$

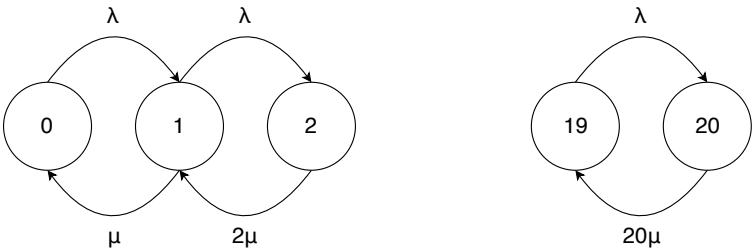


Figure 1: Transition diagram

- The transition diagram is shown in Figure 1.
- Define waiting time $T = T_2 + T_1$. T_2 is exponentially distributed with parameter 2μ, while T_1 is exponentially distributed with parameter μ. The density of T is (see "1.2.5 Sums and convolutions" for this formula and when it is used)

$$f_T(t) = \int_0^t f_{T_2}(s) f_{T_1}(t-s) ds = 2\mu e^{-\mu t} - 2\mu e^{-2\mu t}.$$

d)

- For a birth-and-death process, the sojourn time in state i is an exponential distribution with rate equal to the sum of the birth rate and the death rate, $\lambda_i + \mu_i$. Given that 16 cars are at the garage, the time until the number of cars changes is then an exponential distribution with rate $16/30 + 0.5 = 0.91$.
- The time departure is exponential with rate 16μ and the time to arrival is exponential with rate λ. So, the probability that the next time the number of cars in the garage changes, it will be a car that leaves:

$$\begin{aligned} P_x(x) &= P(\text{Time departure} < \text{Tine arrival}) \\ &= \frac{16\mu}{16\mu + \lambda} \\ &= \frac{16/30}{0.5 + 16/30} \\ &= 0.516 \end{aligned}$$