TMA4265 Stochastic Modelling: Solution Exercise 7

Week 42

Problem 1

a)

By considering the Kolmogorov forward equations as time goes to infinity, we end up with the equations

$$\begin{aligned}\mu_1\pi_1 =&\lambda_0\pi_0\\ \mu_{j+1}\pi_{j+1} =&(\lambda_j+\mu_j)\pi_j-\lambda_{j-1}\pi_{j-1},\ 1\leq j\leq 19\\ \mu_{20}\pi_{20} =&\lambda_{19}\pi_{19},\end{aligned}$$

(see chapter 6.4 of Pinsky and Karlin). In last weeks exercise we found that $\lambda_i=\lambda$ and $\mu_i=i\mu$, since there are i cars that can leave, all with the same rate μ. From chapter 6.4 of Pinsky and Karlin we find that the solution to these equations are given by

$$\pi_j=\theta_j\pi_0,\quad \text{with}\quad \theta_j=\frac{\prod_{i=0}^{j-1}\lambda_i}{\prod_{i=1}^{j}\mu_i}=\frac{\lambda^j}{j!\mu^j}.$$

This gives

$$\pi_j=\frac{\lambda^j}{j!\mu^j}\pi_0,\ j=1,\ldots,20.$$

Thus, if we can find π_0, then all the other limiting probabilities will be found. We use the fact that all the limiting probabilities must sum to one:

$$\sum_{j=0}^{20}\pi_j=1=\pi_0\sum_{j=0}^{20}\frac{\lambda^j}{j!\mu^j}\implies \pi_0=\frac{1}{\sum_{j=0}^{20}\frac{\lambda^j}{j!\mu^j}}.$$

In order to simplify this expression, we use the fact that the cumulative distribution function of the Poisson distribution with parameter ν is equal to

$$F(x,\nu)=\sum_{j=0}^{x}\frac{\nu^j}{j!}e^{-\nu},$$

so if we set $\nu=\lambda/\mu$, then we can write

$$\pi_0=\frac{1}{\sum_{j=0}^{20}\frac{\nu^j}{j!}}=\frac{1}{F(20,\nu)e^{\nu}}.$$

Thus, the limiting probabilities are

$$\pi_j=\frac{\nu^j}{j!}\pi_0=\frac{\nu^j}{j!}\frac{1}{e^{\nu}F(20,\nu)},$$

which we can further simplify as

$$\pi_j = \frac{\frac{\nu^j}{j!}e^{-\nu}}{F(20,\nu)} = \frac{f(j,\nu)}{F(20,\nu)},$$

where $f(x,\nu)$ is the probability mass function of a Poisson process with parameter ν.

In order to compute the numerical value of π_{20} we compute the fraction $\nu = \lambda/\mu = 15$, and from "Tabeller of formler i statistikk'' we get $F(20,15) \approx 0.9170$, and $F(19,15) \approx 0.8752$. This gives

$$\pi_{20} = \frac{f(20,15)}{F(20,15)} = \frac{F(20,15) - F(19,15)}{F(20,15)} \approx \frac{0.9170 - 0.8752}{0.9170} \approx 0.0456.$$

b)

The long-run mean number of cars, N, in the parking garage is equal to

$$E[N] = \sum_{i=0}^{20} i\pi_i = \sum_{i=0}^{20} i\frac{\nu^i}{i!}\pi_0 = \pi_0 \sum_{i=1}^{20} i\frac{\nu^i}{i!},$$

since the contribution from $i = 0$ is 0. By multiplying i with $1/i!$ we further get

$$\pi_0 \sum_{i=1}^{20} i\frac{\nu^i}{i!} = \pi_0\nu \sum_{i=0}^{19} \frac{\nu^i}{i!} = \pi_0\nu \sum_{i=0}^{19} f(i,\nu)e^{\nu},$$

which can be written as

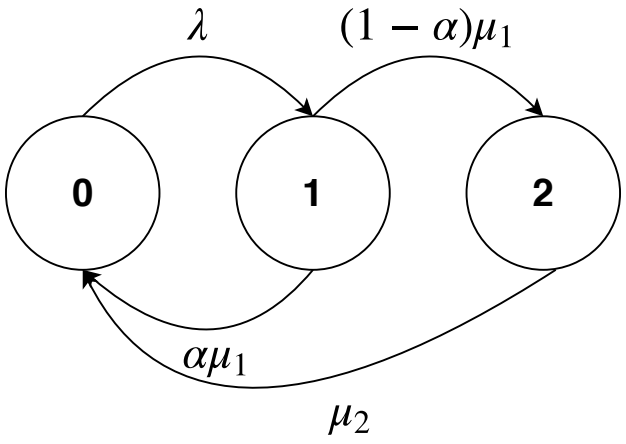
$$\pi_0\nu \sum_{i=0}^{19} f(i,\nu)e^{\nu} = \pi_0\nu e^{\nu} F(19,\nu).$$

Inserting the expression for π_0 gives

$$E[N] = \pi_0\nu e^{\nu} F(19,\nu) = \frac{\nu \exp(\nu) F(19,\nu)}{\exp(\nu) F(20,\nu)} \approx 15\frac{0.8752}{0.9170} \approx 14.32.$$

Problem 2

a) Let us assume we have three states: 0 (guide is free), 1 (guide is negotiating), 2 (guide is on tour). The diagram is



and the balance equations are:

$$\begin{aligned} \text{leaving rate} &= \text{arrival rate} \\ \lambda\pi_0 &= \alpha\mu_1\pi_1 + \mu_2\pi_2 \\ \mu_1\pi_1 &= \lambda\pi_0 \\ \mu_2\pi_2 &= (1-\alpha)\mu_1\pi_1. \end{aligned}$$

Using the fact that the probabilities must sum to one,

$$\pi_0 + \pi_1 + \pi_2 = 1,$$

then substituting the expressions above, we get:

$$\pi_0 + \frac{\lambda}{\mu_1}\pi_0 + \frac{(1-\alpha)\lambda}{\mu_2}\pi_0 = 1$$

Then finally, the proportion of time that the guide is free is:

$$\pi_0 = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1(1-\alpha)}$$

b)

If the guide is currently negotiating with a potential customer, then either they don't reach an agreement and the guide is free, or they agree and go on a tour and after that the guide is free. We know that the mean waiting time in state 1 is μ_1^{-1}. After that, the wait is either over or we have to wait for the tour to finish, with mean waiting time μ_2^{-1}. First step analysis gives that the weighting time w_1 for going from state 1 to state 0 is equal to

$$w_1 = \frac{1}{\mu_1} + \alpha \cdot 0 + (1-\alpha)\frac{1}{\mu_2} = \frac{1}{\mu_1} + \frac{1-\alpha}{\mu_2}.$$

c)

In the long run, the probability that the guide is free equals π_0. Thus, if a customer arrives at some large time, the probability that they will stop to negotiate with the guide is π_0. The probability that they will further end up going on a tour is $1-\alpha$. Consequently, the probability that customer nr. n goes on a tour as n approaches infinity is equal to $\pi_0(1-\alpha)$, meaning that in the long run, $\pi_0(1-\alpha)$ of the arriving customers will go on a tour.

d)

```
lambda = 10
mu_1 = 1 / 5
mu_2 = 1 / 30
alpha = 0.2


# ------------------
# a)
# ------------------
max_time = 50000
time = 0
time_in_states = rep(0, 3)

set.seed(1)
while (time < max_time) {
```

```
  # Assume that we start in state 0
  time_until_customer = rexp(1, lambda)
  time_in_states[1] = time_in_states[1] + time_until_customer

  time_negotiating = rexp(1, mu_1)
  time_in_states[2] = time_in_states[2] + time_negotiating

  agreement_reached = sample(c(TRUE, FALSE), 1, prob = c(1 - alpha, alpha))

  if (agreement_reached) {
    time_on_tour = rexp(1, mu_2)
    time_in_states[3] = time_in_states[3] + time_on_tour
  }

  time = sum(time_in_states)
}

# Estimated limiting probabilities
time_in_states / time
```

```
## [1] 0.003190851 0.173123274 0.823685876
```

```
# True limiting probability for state 0:
mu_1 * mu_2 / (mu_1 * mu_2 + lambda * mu_2 + lambda * mu_1 * (1 - alpha))
```

```
## [1] 0.003436426
```

```
# ------------------
# b)
# ------------------
N = 1000
waiting_times = rep(0, N)

set.seed(1)
for (i in 1:N) {
  # We start in state 1
  time_negotiating = rexp(1, mu_1)
  waiting_times[i] = waiting_times[i] + time_negotiating

  agreement_reached = sample(c(TRUE, FALSE), 1, prob = c(1 - alpha, alpha))

  if (agreement_reached) {
    time_on_tour = rexp(1, mu_2)
    waiting_times[i] = waiting_times[i] + time_on_tour
  }
}

# Estimated mean waiting time
mean(waiting_times)
```

```
## [1] 28.80198
```

```
# True mean waiting time
1 / mu_1 + (1 - alpha) / mu_2
```

```
## [1] 29
```

```
# ------------------
# c)
# ------------------
max_time = 10000

# First we sample all the times that a potential customer arrive
set.seed(1)
customer_arrival_times = rexp(1, lambda)
i = 1
while (customer_arrival_times[i] < max_time) {
  customer_arrival_times[i + 1] = customer_arrival_times[i] + rexp(1, lambda)
  i = i + 1
}

time = 0
num_customers_on_tour = 0
while (time < max_time) {
  # Assume that we start in state 0
  next_customer = which(customer_arrival_times >= time)[1]
  next_customer_time = customer_arrival_times[next_customer]

  time_negotiating = rexp(1, mu_1)
  time = next_customer_time + time_negotiating

  agreement_reached = sample(c(TRUE, FALSE), 1, prob = c(1 - alpha, alpha))

  if (agreement_reached) {
    time_on_tour = rexp(1, mu_2)
    time = time + time_on_tour
    num_customers_on_tour = num_customers_on_tour + 1
  }
}

num_potential_customers = length(customer_arrival_times)

# Estimated proportion
num_customers_on_tour / num_potential_customers
```

```
## [1] 0.003008062
```

```
# True proportion
mu_1 * mu_2 / (mu_1 * mu_2 + lambda * mu_2 + lambda * mu_1 * (1 - alpha)) * (1 - alpha)
```

```
## [1] 0.002749141
```

Problem 3

a)

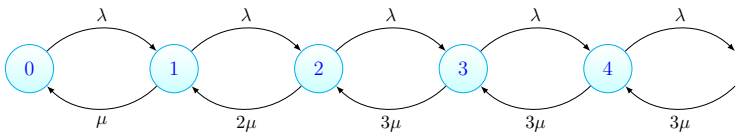
The given process is a birth and death process as the number of skiers in the stadium either increase with one (birth) or decrease with one (death) and all times until the next arrival (birth) and termination of shooting (death) are independent and exponentially distributed. The birth rates are given by

$$\lambda_n = \lambda, \quad n = 1, 2, \dots$$

while the death rates are

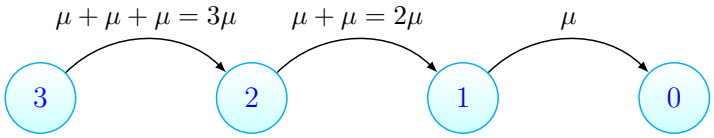
$$\mu_1 = \mu,\ \mu_2 = 2\mu,\ \mu_3 = 3\mu,\ \mu_4 = 3\mu,\ \mu_5 = 3\mu,\ \ldots$$

This is a M/M/3 queue and the transition matrix is\



b)

The time until a skier is finished with shooting is exponentially distributed with rate $\mu = 1/30$. The time until the next skier is finished is then the minimum of independent and exponentially distributed variables which is exponential with rate equal to the sum of the individual rates. Meaning that either of the three completions, assuming $X(t) = 3$ will move the system to $X(t) = 2$, and so on\



The expected time W until all the of the three skiers are finished is

$$E(W) = \frac{1}{3\mu} + \frac{1}{2\mu} + \frac{1}{\mu} \approx 55 \text{ seconds.}$$

c)

Let $t_{i,j}$ denote the expected time it takes before there are j skiers in the stadium, given that we started with i skiers in the stadium. A first step analysis gives that the expected time for going from i to $i+1$ skiers equals

$$t_{i,i+1} = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} t_{i-1,i+1},$$

since the mean waiting time in state i is $(\lambda_i + \mu_i)^{-1}$, and the probability for going from i to $i-1$ is $\mu_i/(\lambda_i + \mu_i)$. In order to go from state $i-1$ to $i+1$, we first have to go from $i-1$ to i and then from i to $i+1$. Thus, we know that $t_{i-1,i+1} = t_{i-1,i} + t_{i,i+1}$. Inserting this into the previous equation gives

$$t_{i,i+1} = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} t_{i-1,i+1} \implies t_{i,i+1} = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} t_{i-1,i}.$$

This gives us the set of equation

$$\begin{aligned} t_{0,1} &= 1/\lambda_0 = 12 \\ t_{1,2} &= 1/\lambda_1 + \frac{\mu_1}{\lambda_1} t_{0,1} = 16.8 \\ t_{2,3} &= 1/\lambda_2 + \frac{\mu_2}{\lambda_2} t_{1,2} = 25.44, \end{aligned}$$

and we finally get that

$$t_{0,3} = t_{0,1} + t_{1,2} + t_{2,3} = 54.24.$$