Solution TMA 4267, Spring 2010

1a)

$$Y = X_1 + X_2 + X_3 = [1, 1, 1]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underline{a}'\underline{X}$$

Hence $$E(Y) = [1,1,1]\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = 4 - 3 + 1 = 2$$

$$Var(Y) = [1,1,1]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1,1,1]\begin{bmatrix} 2 \\ -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix} = 2 - \frac{1}{2} + \frac{7}{2} = 5$$

$Y = \underline{a}'\underline{X}$, where $\underline{X} \sim N_3(\underline{\mu}, \underline{\Sigma})$. Hence $Y \sim N(2, 5)$

The distribution of $X_2 | X_1 = x_1, X_3 = x_3$ is the distribution of

$X_2 | X_3 = x_3$. $\quad X_2 | X_3 = x_3 \sim N\left(\mu_2 + \frac{\sigma_{23}}{\sigma_{33}}(x_3 - \mu_3), \ \sigma_{22} - \frac{\sigma_{23}^2}{\sigma_{33}}\right)$

Hence $X_2 | X_3 = x_3 \sim N\left(-3 - \frac{3}{2 \cdot 5}(x_3 - 1), \ 1 - \frac{9}{4.5}\right)$

i.e. $N\left(-\frac{27}{10} - \frac{3}{10}x_3, \ \frac{11}{20}\right)$

b)
$$|\underline{\Sigma} - \lambda\underline{I}| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5-\lambda \end{vmatrix} = (2-\lambda)\left((1-\lambda)(5-\lambda) - \frac{9}{4}\right)$$

$$= (2-\lambda)\left(5 - 5\lambda - \lambda + \lambda^2 - \frac{9}{4}\right) = (2-\lambda)\left(\lambda^2 - 6\lambda + \frac{11}{4}\right) = 0 \Rightarrow \begin{cases} \lambda = 2 \\ \lambda = \frac{11}{2} \\ \lambda = \frac{1}{2} \end{cases}$$

$\lambda = 2$
$$\underline{\Sigma}\,\underline{x} = 2\underline{x} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Leftrightarrow \begin{aligned} 2x_1 &= 2x_1 \\ x_2 - \tfrac{3}{2}x_3 &= 2x_2 \\ -\tfrac{3}{2}x_2 + 5x_3 &= 2x_3 \end{aligned} \qquad \text{a solution is } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\underline{\lambda = \frac{11}{2}}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{11}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \iff \begin{array}{l} 4x_1 = 11x_1 \\ 3x_2 = -x_3 \end{array}$$

A solution is: $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

$\underline{\lambda = \frac{1}{2}}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \iff \begin{array}{l} 4x_1 = x_1 \\ x_2 = 3x_3 \end{array}$$

a solution is $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

2a). $H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$ $\qquad H_1:$ at least one $\alpha \neq 0$

$$F_{obs} = \frac{SS_T/2}{SS_E/9} = 7.56$$

$P(F_{2,9} \geq 7.56) = 0.012 < 0.05$ i.e. reject H_0 at a 5% level.

$\hat{\alpha}_1 = -23.75 - (-12.58) = -23.75 + 12.58 = -11.17$

$\hat{\alpha}_2 = -14.5 - (-12.58) = -14.5 + 12.58 = -1.92$

$\hat{\alpha}_3 = 0.5 - (-12.58) = 0.5 + 12.58 = 13.08$

Pairwise comparison using Tukey gives that healthy diet and training is significantly different from no intervention.

b)

$H_0: \beta_1 = \beta_2 = \beta_3 = 0$ $\quad$ H_1: at least one is different from zero.

The p-value of the F-statistic is 0.0058, i.e. Ho is rejected.

$$SS_E = (7.34)^2 \cdot 8 = \sum_{i=1}^{12}(y_i - \hat{y}_i)^2 = 431.005$$

$$\sum_{i=1}^{12}(\hat{y}_i - \bar{y}_i)^2 = SS_R, \quad \frac{SS_R/3}{SS_E/8} = 9.16 \Rightarrow SS_R = \frac{SS_E}{8} \cdot 9.16 \cdot 3 = (7.34)^2 \cdot 9.16 \cdot 3 = 1480.51$$

$$\sum_{i=1}^{12}(y_i - \bar{y})^2 = SS_T = SS_R + SS_E = (7.34)^2(8 + 9.16 \cdot 3) = 1911.51$$

2c)

$H_0: \beta_1 = 0$ $\quad$ $H_1: \beta_1 \neq 0$

$$T = \frac{\hat{\beta}_1}{\hat{S}_{\hat{\beta}_1}}. \quad T_{obs} = 1.19$$

$P(T_8 \geq T_{obs}) = 0.27$ i.e. Ho is not rejected

$$F = \frac{\frac{SS_E(R) - SS_E(F)}{1}}{\frac{SS_E(F)}{8}} = \frac{(7.513)^2 \cdot 9 - (7.339)^2 \cdot 8}{(7.339)^2} = 1.432 = (1.19)^2.$$

$f_{0.05,1,8} = 5.132$ i.e Ho is not rejected.

The model indicates that the reduction in blood pressure depends on the BMI-indeks ~~[illegible]~~ before treatment. The larger BMI, the more reduction. The BMI is proportional to your weight, i.e. the more you weigh the more is possible to gain with a healthy diet and training.

3a) i) $Y = X^{\beta} \cdot \varepsilon$ is nonlinear in β

$\ln Y = \beta \ln x + \ln \varepsilon$ is linear ; $Y' = \ln Y$ and $x' = \ln x$

ii) $Y = \alpha + \beta\sqrt{x} + \varepsilon$ is a linear model in β. $Y' = Y$, $x' = \sqrt{x}$

iii) $Y = \dfrac{X}{\alpha + (\beta + \varepsilon)X}$ is nonlinear in α and β

$\dfrac{1}{Y} = \dfrac{\alpha}{X} + \beta + \varepsilon$ is linear, $Y' = \frac{1}{Y}$, $X' = \frac{1}{X}$

b) $\underline{H}^2 = \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t = \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t = H$

$(\underline{I}-H)(\underline{I}-H) = \underline{I}^2 - \underline{H}\underline{I} - \underline{I}\underline{H} + \underline{H}^2 = \underline{I} - \underline{H} - \underline{H} + \underline{H}^2 = \underline{I} - \underline{H}$

$\hat{Y} = \underline{H}\underline{Y}$

$Y - \hat{Y} = (\underline{I} - \underline{H})\underline{Y}$

$\operatorname{Cov}\begin{bmatrix} \underline{H}\underline{Y} \\ (\underline{I}-\underline{H})Y \end{bmatrix} = E\left[\begin{bmatrix} \underline{H}(\underline{Y}-\underline{X}\beta) \\ (\underline{I}-\underline{H})(\underline{Y}-\underline{X}\beta) \end{bmatrix} \left[(\underline{Y}-\underline{X}\beta)^t H, \; (\underline{Y}-\underline{X}\beta)^t(\underline{I}-\underline{H})\right]\right]$

$$= \begin{bmatrix} \underline{H}\, E[(\underline{Y}-\underline{X}\beta)(\underline{Y}-\underline{X}\beta)^t]\underline{H} & \underline{H}\, E[(\underline{Y}-\underline{X}\beta)(\underline{Y}-\underline{X}\beta)^t](\underline{I}-H) \\ (\underline{I}-\underline{H})\, E[(\underline{Y}-\underline{X}\beta)(\underline{Y}-\underline{X}\beta)^t]\underline{H} & (\underline{I}-\underline{H})\, E[(\underline{Y}-\underline{X}\beta)(\underline{Y}-\underline{X}\beta)^t][\underline{I}-\underline{H}] \end{bmatrix}$$

with $E[(\underline{Y}-\underline{X}\beta)(\underline{Y}-\underline{X}\beta)^t] = \sigma^2\underline{I}$

$$= \begin{bmatrix} \sigma^2\underline{H} & 0 \\ 0 & \sigma^2(\underline{I}-\underline{H}) \end{bmatrix}$$ since $(\underline{I}-\underline{H})\underline{H} = \underline{H} - \underline{H} = 0$

$\underline{\varepsilon} \sim N(0, \sigma^2\underline{I})$, than gives that $\underline{\hat{Y}}$ and $\underline{Y} - \underline{\hat{Y}}$ are independent.

c) $SS_E = \underline{Y}^t(\underline{I}-\underline{H})\underline{Y} = (\underline{Y}-\underline{X}\beta)^t(\underline{I}-\underline{H})(\underline{Y}-\underline{X}\beta)$

Since $(\underline{I}-\underline{H})\underline{X}\beta = \underline{X}\beta - \underline{X}\beta = 0$

$\underline{Y} - \underline{X}\beta = \underline{\varepsilon}$ and $\underline{\varepsilon} \sim N(\underline{0}, \sigma^2\underline{I})$. Rank $(\underline{I}-\underline{H}) = tr(\underline{I}) - tr(\underline{H}) = n-(k+1)$

Hence $SSE/\sigma^2 \sim \chi^2(n-k-1)$.

$(H - \frac{1}{n}J)(I-H) = H - \frac{1}{n}J - H + \frac{1}{n}J = 0$. Hence $\frac{SSE}{\sigma^2}$ and $\frac{SSR}{\sigma^2}$ are independent

and $\dfrac{SSR}{\sigma^2(k)} \Big/ \dfrac{SSE}{\sigma^2(n-k+1)}$ are F-distributed $k, n-k-1$ since rank$(\underline{H} - \frac{1}{n}\underline{J}) = k+1-1 = k$.

and $(H - \frac{1}{n}J)(H - \frac{1}{n}J) = H^2 - \frac{1}{n}JH - \frac{1}{n}JH + \frac{1}{n^2}JJ = H - \frac{2}{n}J + \frac{1}{n}J = H - \frac{1}{n}J$