Solution TMA 4267 V-2011

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a) $\underline{Y} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underline{A}\,\underline{X}$ and therefore binormal-distributed since $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is binormal distributed.

$$\mu_{\underline{Y}} = \begin{bmatrix} \mu(1+k) \\ \mu \end{bmatrix}. \qquad \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{Cov(X_1,X_2)}{\sigma k \sigma} = \rho \Rightarrow Cov(X_1,X_2) = \rho k \sigma^2.$$

$$\Sigma_{\underline{X}} = \sigma^2 \begin{bmatrix} 1 & \rho k \\ \rho k & k^2 \end{bmatrix} \Rightarrow \Sigma_{\underline{Y}} = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \rho k \\ \rho k & k^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1+2\rho k + k^2 & 1+\rho k \\ 1+\rho k & 1 \end{bmatrix}$$

b) $$E[X_1 + X_2 \mid X_1 = x_1] = \mu(1+k) + \frac{\sigma^2(\rho k+1)}{\sigma^2}(x_1 - \mu)$$

$$= \mu(1+k) + (\rho k+1)(x_1-\mu)$$

$k = 4,\ \rho = \frac{1}{2},\ x_1 - \mu = 1 \Rightarrow \quad E[X_1+X_2 \mid X_1 = \mu+1] = \quad 5\mu + 3$

$k = 1$ og $\rho = \frac{1}{2} \Rightarrow \quad \Sigma_{\underline{X}} = \sigma^2 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda' \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \Rightarrow \begin{aligned} e_1 + \tfrac{1}{2} e_2 &= \lambda' e_1 \quad (1) \\ \tfrac{1}{2} e_1 + e_2 &= \lambda' e_2 \quad (2) \end{aligned} \quad , \quad \lambda' = \frac{\lambda}{\sigma^2}$$

(1) $\Rightarrow$ $e_1 = \dfrac{e_2}{2(\lambda'-1)}$ which inserted in (2) gives

$$\frac{e_2}{4(\lambda'-1)} + e_2 = \lambda' e_2 \Rightarrow \lambda'^2 - 2\lambda' + \frac{3}{4} = 0 \Rightarrow \lambda' = \begin{cases} \frac{3}{2} \\ \frac{1}{2} \end{cases} \Rightarrow \lambda = \begin{cases} \frac{3\sigma^2}{2} \\ \frac{\sigma^2}{2} \end{cases}$$

$\lambda' = \frac{3}{2} \Rightarrow e_2 = e_1$ and the 1. principal component is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

which explain $\dfrac{\frac{3}{2}}{2} = 75\%$ of $Var(X_1 + X_2]$

2a). $$F_{obs} = \frac{\frac{SS_R}{3}}{\frac{SS_E}{14}} = 69.12.$$

$P(F_{3,14} \geq 69.12) = 1.236 \cdot 10^{-8}$ shows that the regression is significant.

Multiple R-squared $= \frac{SS_R}{SS_T}$ and tells how much of the variation in the response that is explained by the model.

$$s = \sqrt{\frac{SS_E}{14}} = 19.63 \Rightarrow SS_E = (19.63)^2 \cdot 14 = 5394.72.$$

From F_{obs} we have $SS_R = \frac{SS_E}{14} \cdot 69.12 \cdot 3 = (19.63)^2 \cdot 69.12 \cdot 3$

$= 79903.46$

b) $$SS_E^2 = (14.59)^2 \cdot 12 = 2554.42.$$

$$E(Y) = \beta_0 + \beta_1 ph + \beta_2 z_1 + \beta_3 z_3 + \beta_4 ph\, z_1 + \beta_5 ph\, z_2$$

$H_0: \beta_3 = \beta_5 = 0$ $\qquad H_1$: minst en $\neq 0$.

$$F_{2,12} = \frac{\frac{SS_R(FM) - SS_R(RM)}{2}}{\frac{SS_E(FM)}{12}} = \frac{\frac{SS_E(RM) - SS_E(FM)}{2}}{\frac{SS_E(FM)}{12}}$$

$$= \frac{\frac{5394.72 - 2554.42}{2}}{\frac{2554.42}{12}} = 6.67 > 3.89 = f_{0.05}(2,12)$$ i.e

at least one of the terms with z_2 should be in the model

c) $E(Y) = \beta_0 + \beta_1 \, ph + \beta_2 \, z1 + \beta_3 \, ph \, z1 + \beta_4 \, ph \, z2$

$H_0: \beta_4 = 0$ $\quad H_1: \beta_4 \neq 0$.

$t_{obs} = 3.16$. $P(|T_{13}| \geq 3.16) = 0.00755$ which implies

that ph z2 is significant at a 5% level given the other variables.

Polymer 3 $\quad z_1 = z_2 = 0$

$$\hat{y} = -208.98 + 60.31 \, ph$$

Polymer 2 $\quad z_1 = 0, \quad z2 = 1$

$$\hat{y} = -208.98 + 60.31 \, ph + 3.58 \, ph$$

$$= -208.98 + 63.89 \, ph.$$

Polymer 1

$$\hat{y} = -208.98 + 60.31 \, ph + 248.395 - 20.05 \, ph$$

$$= 39.415 + 40.16 \, ph.$$

3a) $\hat{\beta} = (X^t X)^{-1} X^t (X\beta + \varepsilon) = \beta + (X^t X)^{-1} X^t \varepsilon$

such that $\hat{\beta} - \beta = (X^t X)^{-1} X^t \varepsilon$

$Cov(\hat{\beta}) = E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t] = E[(X^t X)^{-1} X^t \varepsilon \varepsilon^t X (X^t X)^{-1}] = \sigma^2 (X^t X)^{-1}$

$e = Y - X\hat{\beta} = Y - X(X^t X)^{-1} X^t Y = (I - H)Y$

$= (I-H)(Y - X\beta) = (I-H)\varepsilon$ since $(I-H)(Y-X\beta) = (I-H)Y - X\beta + X\beta$

$= (I-H)Y$

b) $SS_E = e^t e = \varepsilon^t (I-H)^t (I-H)\varepsilon = \varepsilon^t (I-H)\varepsilon$ since $(I-H)$

is symmetric and idempotent.

$E[SS_E] = tr[(I-H)\cdot \sigma^2 I] + 0^t (I-H) 0 = \sigma^2 tr(I-H)$

$= \sigma^2 (tr(I) - tr(H)) \overset{H \text{ idempotent}}{=} \sigma^2 (n - rank(H)) = \sigma^2 (n - (k+1))$

$\hat{\sigma}^2 = SSE/(n-k-1)$

c) $\dfrac{SSE}{\sigma^2} = \dfrac{\varepsilon^t (I-H)\varepsilon}{\sigma^2} \sim \chi^2(rank(I-H))$ since $\varepsilon \sim N(0, \sigma^2 I)$

$rank(I-H) = tr(I-H)$ since $I-H$ is idempotent.

$tr(I-H) = n-k-1.$

$Cov(\hat{\beta}, e) = E[(X^t X)^{-1} X^t \varepsilon \varepsilon^t (I-H)] = (X^t X)^{-1} X^t \sigma^2 I (I-H)$

$= \sigma^2 (X^t X)^{-1} X^t (I-H) = \sigma^2 ((X^t X)^{-1} X^t - (X^t X)^{-1} X^t X (X^t X)^{-1} X^t] = \sigma^2 0$

Since $\hat{\beta}$ and e are independent it follows that $\hat{\beta}$ and $e^t e$ also are independent.