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Solution TMA 4267 August 2013

Problem 1

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \underline{A}\,\underline{V}$$

$$E\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Cov}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $\begin{bmatrix} U \\ V \end{bmatrix}$ is a linear transformation of $\underline{V}$ which is trivariate normal distributed $\begin{bmatrix} U \\ V \end{bmatrix}$ is bivariate normal distributed i.e. $\begin{bmatrix} U \\ V \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\right)$

$$\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & a & b \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \Rightarrow E\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Cov}\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & a & b \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 4 & 2a-b \\ 2a-b & 2a^2-2ab+b^2 \end{bmatrix}$$

Hence the correlation between U and W is zero when $b = 2a$. Since $\begin{bmatrix} U \\ W \end{bmatrix}$ is bivariate normal distributed they are also independent when $b = 2a$.

Problem 2

a)

A two-way analysis of variance has been performed

α_i, $i = 1, 2, 3$ is the main effect of the i-th level of sand

β_j, $j = 1, 2, 3$ is the main effect of the j-th level of carbon

$(\alpha\beta)_{ij}$ is the interaction effect between the i-th level og sand and j-th level of carbon, $i = 1, 2, 3$, $j = 1, 2, 3$.

1. H_0^1: $\alpha_1 = \alpha_2 = \alpha_3 = 0$, H_1^1: at least one $\alpha_i \neq 0$, $i = 1, 2, 3$

2. H_0^2: $\beta_1 = \beta_2 = \beta_3 = 0$, H_1^2: at least one $\beta_j \neq 0$, $j = 1, 2, 3$

3. H_0^3, $(\alpha\beta)_{11} = (\alpha\beta)_{12} = \cdots (\alpha\beta)_{33} = 0$, H_1^3: at least one $(\alpha\beta)_{ij} \neq 0$, $i = 1, 2, 3$ $j = 1, 2, 3$.

b)

Let
$$SS_E = \sum_{k=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(Y_{ijk} - \bar{Y}_{ij\cdot})^2$$
$$SS_S = 6\sum_{i=1}^{3}(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$
$$SS_C = 6\sum_{j=1}^{3}(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$
$$SS_{SC} = 2\sum_{i=1}^{3}\sum_{j=1}^{3}(\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot})^2$$

Then the respective test statistics are

1: $F_1 = \dfrac{SS_S/2}{SS_E/9}$, 2: $F_2 = \dfrac{SS_C/2}{SS_E/9}$, 3: $F_3 = \dfrac{SS_{SC}/4}{SS_E/9}$

$$P(F_1 \geq F_{1,obs} = 6.54) = 0.018$$
$$P(F_2 \geq F_{2,obs} = 5.33) = 0.03$$
$$P(F_3 \geq F_{3,obs} = 0.27) = 0.89$$
$\Rightarrow$ In this case only the main effects are significant on a 5% level.

Hypothesis 3 should always be performed before the two others.

c)

$$\hat{\alpha}_1 = \bar{Y}_{1..} - \bar{Y}_{...} = 66.333 - 69.611 - \underline{3.28}$$

$$\hat{\beta}_2 = \bar{Y}_{.2.} - \bar{Y}_{...} = 71.17 - 69.61 = \underline{1.56}$$

$$(\hat{\alpha}\hat{\beta})_{33} = \bar{Y}_{33.} - \bar{Y}_{3..} - \bar{Y}_{.3.} + \bar{Y}_{...} = 74 - 72.17 - 71.17 + 69.61 = \underline{0.27}$$

Let $\mu_{33} = E[\bar{Y}_{33.}]$ and $\mu_{11} = E[\bar{Y}_{11.}]$

then $\bar{Y}_{33.} \sim N(\mu_{33}, \frac{\sigma^2}{2})$ and $\bar{Y}_{11.} \sim N(\mu_{11}, \frac{\sigma^2}{2})$

and $\bar{Y}_{33.} - \bar{Y}_{11.} \sim N(\mu_{33} - \mu_{11}, \sigma^2)$

$H_0: \mu_{33} = \mu_{11}$ $\qquad H_1: \mu_{33} > \mu_{11}$

Test statistics $T = \frac{\bar{Y}_{33.} - \bar{Y}_{11.}}{S} \sim$ t-distributed with 9 degrees of freedom

Reject if $P(T_9 \geq \frac{12}{\sqrt{8.16}}) \leq 0.05$

$P(T_9 \geq 4.2) < 0.005 \Rightarrow$ reject. i.e The hardness on the level combination (S3, C3) is harder

d)

The coefficient in front of X_2^2 in model 2 is not significant on a 5% level. However if we compare the R^2, R^2_{adj} and p-values for significant regression we get.

Model	R^2	R^2 adj	p-values
Model 1	0.61	0.55	0.00093
Model 2	0.68	0.62	0.00083

Hence in this case it is natural to prefer model 2.

The column for X_{22} is $(1,1,1,1,1,1,0,0,0,0,0,0,1,1,1,1,1,1)$ which is orthogonal to both $\underline{x}_1$ and $\underline{x}_2$. Hence the estimation of the coefficients in front of x_1 and x_2 are not affected by adding the term x_{22} to the model.

Problem 3

a) $$\hat{\underline{\mu}} = \underline{X}\hat{\underline{\beta}}$$

$$E[\hat{\underline{\mu}}] = E[\underline{X}\hat{\underline{\beta}}] = \underline{X}\,E[\hat{\underline{\beta}}] = \underline{X}\underline{\beta}$$

$$Cov[\hat{\underline{\mu}}] = \underline{X}\,Cov(\hat{\underline{\beta}})\,\underline{X}^t = \sigma^2 \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t = \sigma^2 \underline{H}$$

b)

$\underline{H}$ is idempotent and symmetric $\Rightarrow$ $rank(\underline{H}) = trace(\underline{H})$

$$\sum_{i=1}^{n} Var(\hat{\mu}_i) = \sigma^2\, trace(\underline{H}) = \sigma^2\, rank(\underline{H}) = \sigma^2(p+1)$$

c)

$$\underline{e} = (\underline{I}-\underline{H})(\beta_0 \underline{1} + \beta_1 \underline{x}_1 + \beta_2 \underline{x}_2 + \underline{\varepsilon})$$

$$= \beta_0(\underline{I}-\underline{H})\underline{1} + \beta_1(\underline{I}-\underline{H})\underline{x}_1 + \beta_2(\underline{I}-\underline{H})\underline{x}_2 + (\underline{I}-\underline{H})\underline{\varepsilon}$$

$\underline{H}\,\underline{1} = \underline{1}$, $\quad \underline{H}\,\underline{x}_1 = \underline{x}_1 \Rightarrow \underline{e} = \beta_2(\underline{I}-\underline{H})\underline{x}_2 + (\underline{I}-\underline{H})\underline{\varepsilon}$

$\underline{e} = (\underline{I}-\underline{H})\underline{Y}$. $\quad E[\underline{e}^t\underline{e}] = E[\underline{Y}^t(\underline{I}-\underline{H})\underline{Y}]$ which is a quadratic form

$Cov(\underline{Y}) = \sigma^2\underline{I}$. $\quad$ Hence $E[\underline{e}^t\underline{e}] = tr((\underline{I}-\underline{H})\sigma^2\underline{I}) + \underline{\mu}^t(\underline{I}-\underline{H})\underline{\mu}$

$tr(\underline{I}-\underline{H}) = n-2$

$(\underline{I}-\underline{H})\underline{\mu} = \beta_2(\underline{I}-\underline{H})\underline{x}_2$

Hence $E[\underline{e}^t\underline{e}] = \sigma^2(n-2) + \beta_2^2\,\underline{x}_2^t(\underline{I}-\underline{H})\underline{x}_2$