Solution TMA 4267, May 25 2013

Problem 1

a)

$$\rho_{X_3, X_1} = \frac{1}{\sqrt{3}\cdot\sqrt{1}} = \frac{\sqrt{3}}{3}$$

$$\rho_{X_3\, X_2} = \frac{-1}{\sqrt{2}\,\sqrt{3}} = -\frac{\sqrt{6}}{6}, \quad \text{Since } \left|\frac{\sqrt{3}}{3}\right| > \left|-\frac{\sqrt{6}}{6}\right|$$

it follow that X_1 have the strongest correlation with X_3 in absolute value

Let $\underset{\sim}{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then $\underline{Z} = \underline{A}\,\underline{X}$, i.e. a linear transformation of $\underline{X}$ which is trivariate normal and thereby bivariate normal distributed since $\underline{A}$ is a 2×3 matrix

$$E[\underline{Z}] = \underline{A}\mu = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_1 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Sigma_Z = \underline{A}\,\underline{\Sigma}\,\underline{A}^t = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

b)

$$\underline{Y} = \begin{bmatrix} e_1^t \\ e_2^t \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underline{A}\,\underline{X} \text{ and is thereby a linear transformation}$$

of $\underline{X} \sim N_3(\underline{\mu}, \underline{\Sigma})$. $\underline{A}$ is a 2×3 matrix

$$\mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(\underline{e}_1^t\underline{X}, \underline{e}_2^t\underline{X}) = E[\underline{e}_1^t(\underline{X}-\mu)(X-\mu)^t\underline{e}_2]$$

$$= \underline{e}_1^t\,\underline{\Sigma}\,\underline{e}_2 = \underline{e}_1^t\lambda\,\underline{e}_2 = \lambda\,\underline{e}_1^t\underline{e}_2 = \underline{0}$$

Since Y_1 and Y_2 are univariate normal and uncorrelated, they are independent.

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{3.88 + 1.65}{6} = 0.921, \quad \text{i.e about } 92.1\,\%$$

of the total variation in $\underline{X}$ can be explained by $\underline{Y}$.

c)

Since $\underline{Y}$ is bivariate normaldistributed we know that D is χ^2-distributed with two degrees of fredom.

If. $D_{obs} \geq \chi^2_{\alpha,2}$ for some α ~~the~~ the product would not pass the control.

Suppose $\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is trivariate normal distributed with mean vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \underline{\Lambda}$

Let $\underline{X} = \underline{P}\underline{U} = [\underline{e}_1, \underline{e}_2, \underline{e}_3]\,\underline{U}$. Then $Cov(\underline{X}) = \underline{P}\,\underline{\Lambda}\,\underline{P}^t = \underline{\Sigma}$

Hence simulate n observation vectors from $\underline{U}$ and let

$\underline{X} = \underline{P}\underline{U} + \underline{\mu}$

Problem 2

$H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$ $\qquad$ $H_1: \alpha_i \neq 0$ for minst ein i, $i = 1, 2, 3$

$F_{obs} = 15.28.$ $\qquad$ $P(F_{2,6} \geq 15.28) = 0.004 < 0.05$. We conclude that ~~the~~ at least one $\alpha_i \neq 0$, i.e. the declination has changed.

There are three pairwise comparisons $\mu_2 - \mu_1$, $\mu_3 - \mu_1$, $\mu_3 - \mu_2$

Only the confidence interval $\mu_3 - \mu_1$ does not cover 0 using the exact Tukey test. Hence we can only conclude that $\mu_3 \neq \mu_1$

The corresponding time span is $1865 - 1669 = 196$ year.

The two others time span are: 111 and 85 respectively, but according to the Tukey test we cannot conclude that 100 year is sufficient to conclude there has been a change though the comparison for $\mu_3 - \mu_2$ is pretty close.

b) $\beta_1 = \mu + \alpha_1$, $\beta_2 = \mu + \alpha_2$, $\beta_3 = \mu + \alpha_3$

$$Cov(\hat{\beta}) = \sigma^2(\underline{X}^t \underline{X})^{-1} = \sigma^2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \sigma^2 \frac{1}{3} I$$

Hence $\hat{\beta}_1$, ~~and~~ $\hat{\beta}_2$ and $\hat{\beta}_3$ are uncorrelated. Since the Y's are normal distributed they are also independent.

c)

$$\underline{H} = \underline{X} \cdot \frac{1}{3} \underline{I} \cdot \underline{X}^t = \frac{1}{3} \underline{X}\,\underline{X}^t = \frac{1}{3} \begin{bmatrix} \underline{J_3} & \underline{0} & \underline{0} \\ 0 & \underline{J_3} & \underline{0} \\ 0 & 0 & \underline{J_3} \end{bmatrix}$$

Since $\underline{H} = \underline{J}^\circ$, it follows that expression 1 and 2 are equal.

All the matrices defined are shown in to be projection matrices in the course. Hence

$$rank(\underline{I} - \tfrac{1}{9}\underline{J}) = tr(\underline{I}) - \tfrac{1}{9} tr(\underline{J}) = 9 - \tfrac{1}{9} \cdot 9 = 8$$

$$rank(\underline{I} - \underline{J}^\circ) = rank(\underline{I} - \underline{H}) = tr(\underline{I}) - tr(\underline{H}) = 9 - \tfrac{1}{3} \cdot 9 = \underline{6}$$

$$rank(\underline{J}^\circ - \tfrac{1}{9}\underline{J}) = rank(\underline{H} - \tfrac{1}{9}\underline{J}) = tr(H) - \tfrac{1}{9} tr(J) = 3 - 1 = 2$$

$$rank(H) = tr(H) = \underline{3}$$

d) According to our theoritical results since $(\underline{I} - \underline{H})(\underline{Y} - \underline{X}\beta) = (\underline{I} - \underline{H})\underline{Y}$, and $Y \sim N(\underline{X}\beta, \sigma^2 I)$ the two quadratic forms are independent if. $(\underline{I} - \underline{H})\underline{H} = \underline{0}$. $(\underline{I} - \underline{H})\underline{H} = \underline{H} - \underline{H}^2 = \underline{H} - \underline{H} = \underline{0}$.

Hence they are independent.

Also: $\frac{(\underline{Y} - \underline{X}\beta)^t(\underline{I} - H)(\underline{Y} - \underline{X}\beta)}{\sigma^2} = \frac{\underline{Y}^t(\underline{I} - \underline{H})\underline{Y}}{\sigma^2}$ is $\chi^2(rank(\underline{I} - H))$, i.e. $\chi^2(6)$

If $\beta_1 = \beta_2 = \beta_3 = 0$. $E(\underline{Y}) = 0$. The Y's are independent and $Var(Y_{ij}) = \sigma^2$, $\forall\ ij$. Hence $\frac{\underline{Y}^t \underline{H} \underline{Y}}{\sigma^2} \sim \chi^2(\text{rank}(H)) \sim \chi^2(3)$

e)

$H_0^1: \mu_1 = \mu_2 = \mu_3$ $\quad$ $H_1^1: \mu_i \neq \mu_j$ for at least one i, j

This is the normal F-test in a one-way anova.

$$F = \frac{SS_A/2}{SS_E/6} = \frac{\underline{Y}^t(\underline{J}^\circ - \frac{1}{9}\underline{J})\underline{Y}/2}{\underline{Y}^t(\underline{I} - \underline{J}^\circ)\underline{Y}/6} \sim F_{2,6}.$$

From theory we know that $\frac{SS_A}{\sigma^2}$ and $\frac{SS_E}{\sigma^2}$ are independent and $\chi^2(\text{rank}(\underline{J}^\circ - \frac{1}{9}\underline{J}))$ and $\chi^2(\text{rank}(\underline{I} - \underline{J}^\circ))$ respectively. Thereby the distribution. From 2a we know that H_0^1 will be rejected.

H_0^2 $\mu_1 = \mu_2 = \mu_3 = 0 \iff \beta_1 = \beta_2 = \beta_3 = 0$, $\quad$ H_1^2: at least one $\mu_i \neq 0$

From the discussion in 2c and 2d we know that under $H_0^2: \mu_1 = \mu_2 = \mu_3 = 0$

$$\frac{\underline{Y}^t \underline{H} \underline{Y}/3}{\underline{Y}^t(\underline{I} - \underline{H})\underline{Y}/6} \sim F_{3,6}.$$

From the output $P(F_{3,6} \geq F_{obs}) \approx 5.482 \cdot e^{-10}$. Hence H_0^2 is rejected.

$SS_R = \underline{y}^t(\underline{H} - \frac{1}{n}\underline{J})\underline{y} = \underline{y}^t(\underline{J}^\circ - \frac{1}{9}\underline{J})\underline{y} = 90.03$ from R-output before 2a

$SS_T = SS_R + SS_E = 90.03 + 17.67 = 107.70$.

$$R^2 = \frac{90.03}{107.70} \approx \underline{0.84}$$