Løysingsforslag TMA 4267 V-2009

1a)

$$z_1 = [1,1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow E[z_1] = [1,1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$$

$$Var(z_1) = [1,1]\begin{bmatrix} 9 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1,1]\begin{bmatrix} 11 \\ 11 \end{bmatrix} = 22$$

$$\underline{y} \sim N_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 & 2 \\ 2 & 9 \end{bmatrix}\right) \Rightarrow z_1 = [1,1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim \underline{N(2,22)}$$

$$z_2 = y_1 + y_2 - 2y_2 = y_1 - y_2 = [1,-1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow E[z_2] = [1,-1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$Var(z_2) = [1,-1]\begin{bmatrix} 9 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1,-1]\begin{bmatrix} 7 \\ -7 \end{bmatrix} = 14$$

$$z_2 = \underline{a}^t \underline{y}, \text{ where } \underline{a}^t = [1,-1] \Rightarrow \underline{z_2 \sim N(0,14)}$$

$$Cov(z_1, z_2) = E[z_1 z_2] - E[z_1]\cdot E[z_2] = E[z_1 z_2] = E[(y_1+y_2)(y_1-y_2)]$$

$$= E(y_1^2 - y_2^2) = E(y_1^2) - E(y_2^2) = (9+1) - (9+1) = \underline{0}$$

2)

$$\hat{\beta}_1^* = (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{y} = (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t (\underline{X}_1 \beta_1 + \underline{X}_2 \beta_2 + \varepsilon)$$

$$= \beta_1 + (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{X}_2 \beta_2 + (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{\varepsilon}$$

$$E[\hat{\beta}_1^*] = \beta_1 + (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{X}_2 \beta_2 + (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t E[\underline{\varepsilon}] = \beta_1 + (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{X}_2 \underline{\beta}_2 \text{ q.e.}$$

$$Cov(\hat{\beta}_1^*] = E[(\hat{\beta}_1^* - E(\hat{\beta}_1^*))(\hat{\beta}_1^* - E(\hat{\beta}_1^*))^t]$$

$$= E[(\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t \underline{\varepsilon}\, \underline{\varepsilon}^t \underline{X}_1 (\underline{X}_1^t \underline{X}_1)^{-1}] = \sigma^2 (\underline{X}_1^t \underline{X}_1)^{-1} \underline{X}_1^t I \underline{X}_1 (\underline{X}_1^t \underline{X}_1)^{-1}$$

$$= \underline{\sigma^2 (\underline{X}_1^t \underline{X}_1)^{-1}}$$

b)

Unbiased if $\underline{X}_1^t \underline{X}_2 = 0$ or all the columns in $\underline{X}_1$ are orthogonal to the columnss in $\underline{X}_2$

3a)

$$D_i = \frac{(\hat{y}_{(i)} - \hat{y})^t(\hat{y}_{(i)} - \hat{y})}{(k+1)\, s^2}.$$

$\hat{y}$ is the fitted model using all observations

$\hat{y}_{(i)}$ is the fitted model if the i-th observation is taken out.

i.e. $(y_i, x_{1i}, \ldots, x_{ki})$ is taken out.

k is the number of regression variables.

s^2 is an estimate for the variance of the response values,

If $D_i > 1$ we say that the i-th observation has large influence.

4a)

$$\sum_{i=1}^{n} y_i^2 = [y_1, y_2, \ldots, y_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y^t I y$$

$$\left(I - \frac{1}{n}J\right)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix} = \begin{bmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{bmatrix}$$

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = [y_1 - \bar{y}, y_2 - \bar{y}, \ldots, y_n - \bar{y}]\begin{bmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{bmatrix} = [y_1, y_2, \ldots, y_n]\begin{bmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{bmatrix}$$

since $[\bar{y}, \bar{y}, \cdots, \bar{y}]\begin{bmatrix} y_1-\bar{y} \\ \vdots \\ y_n-\bar{y}\end{bmatrix} = \bar{y}\sum_{i=1}^{n}(y_i-\bar{y}) = 0$

Hence $\sum_{i=1}^{n}(y_i-\bar{y})^2 = y^t(I-\frac{1}{n}J)y$

$n\bar{y}^2 = \sum_{i=1}^{n} y_i\bar{y} = [y_1, y_2, \cdots, y_n]\begin{bmatrix}\bar{y}\\ \bar{y}\\ \vdots \\ \bar{y}\end{bmatrix} = y^t(\frac{1}{n}J)y$

b)

1. $(I-\frac{1}{n}J)(I-\frac{1}{n}J) = I^2 - \frac{1}{n}J - \frac{1}{n}J + \frac{1}{n^2}J\cdot J$

$= I - \frac{1}{n}J - \frac{1}{n}J + \frac{1}{n^2}\cdot nJ = I - \frac{1}{n}J$

$\frac{1}{n}J\circ\frac{1}{n}J = \frac{1}{n^2}JJ = \frac{1}{n^2}nJ = \underline{\frac{1}{n}J}$

2. $(I-\frac{1}{n}J)(\frac{1}{n}J) = \frac{1}{n}J - \frac{1}{n^2}\cdot J\cdot J = \frac{1}{n}J - \frac{1}{n}J = 0$

c)

5.5 $y \sim N(0, \sigma^2 I)$, A symmetric with rank r. Then

$y^tAy \sim \chi^2(r) \iff \sigma^2 A$ is idempotent.

Choose $A = \frac{(I-\frac{1}{n}J)}{\sigma^2}$. rank $[I-\frac{1}{n}J] = tr(I) - tr(\frac{1}{n}J)$

$= n-1$

Hence $\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{\sigma^2} = \frac{y^t(I-\frac{1}{n}J)y}{\sigma^2} \sim \chi^2(n-1)$

$\frac{n\bar{y}^2}{\sigma^2} = \frac{y^t(\frac{1}{n}J)y}{\sigma^2} \sim \chi^2(1)$.

3. Choose $B = \frac{\sqrt{n}}{\sigma} j^t$. $\quad By = \frac{\sqrt{n}}{\sigma}\bar{y}$

$\Sigma = \sigma^2 I$ and $B\Sigma A = \frac{\sqrt{n}}{\sigma} \cdot j^t(I - \frac{1}{n}J)$

$= \frac{\sqrt{n}}{n} \cdot (j^t - j^t) = 0$.

Hence. $\sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{\sigma^2}$ and $\frac{\sqrt{n}}{\sigma}\bar{y}$ are independent and

thereby $\sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{\sigma^2}$ and $\frac{n}{\sigma^2}\bar{y}^2$.

Also. You can use R6 and R7 to solve this more directly.

See also Exercise 2. Problem 1 for one possible solution.

$H_0: \mu = 0, \qquad H_1: \mu \neq 0.$

d) $$\frac{\frac{n\bar{y}^2}{\sigma^2}}{\frac{\sum(y_i - \bar{y})^2}{\sigma^2 \, n-1}} \sim \frac{\frac{\chi^2(1)}{1}}{\frac{\chi^2(n-1)}{n-1}} \sim F_{1,n-1}$$ since $\frac{n\bar{y}^2}{\sigma^2}$ and

$\frac{\sum(y_i - \bar{y})^2}{\sigma^2}$ are independent.

e) $$T = \frac{\bar{y}}{s/\sqrt{n}} = \frac{\sqrt{n}\,\bar{y}}{\sqrt{\frac{\sum(y_i - \bar{y})^2}{n-1}}}. \qquad T^2 = \frac{n\bar{y}^2}{\frac{\sum(y_i - \bar{y})^2}{n-1}} = F_{1,n-1}$$