Solution TMA 4267 23/5 - 2012

Problem 1

a)

$$\text{Cov}(X_1, X_2) = \rho\, SD(X_1)\, SD(X_2) = \rho \sqrt{k\mu_1^2}\sqrt{k\mu_2^2} = \rho k \mu_1 \mu_2$$

$$\Sigma_{X_1,X_2} = \begin{bmatrix} k\mu_1^2 & \rho k \mu_1 \mu_2 \\ \rho k \mu_1 \mu_2 & k\mu_2^2 \end{bmatrix}$$

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Rightarrow \underline{\mu}_Y = E\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 - \mu_1 \end{bmatrix}$$

$$\Sigma_{Y_1Y_2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k\mu_1^2 & \rho k \mu_1 \mu_2 \\ \rho k \mu_1 \mu_2 & k\mu_2^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k\mu_1^2 & \rho k \mu_1\mu_2 - k\mu_1^2 \\ \rho k \mu_1 \mu_2 & k\mu_2^2 - \rho k \mu_1 \mu_2 \end{bmatrix} = \begin{bmatrix} k\mu_1^2 & \rho k \mu_1 \mu_2 - k\mu_1^2 \\ \rho k \mu_1 \mu_2 - k\mu_1^2 & k\mu_2^2 - 2\rho k \mu_1 \mu_2 + k\mu_1^2 \end{bmatrix}$$

$$\underline{Y} \sim N_2(\underline{\mu}_Y, \Sigma_{Y_1,Y_2})$$

b) $Y_1 = X_1$

$$Y_2 \mid X_1 = x_1 \sim N\left(\mu_2 - \mu_1 + \frac{\rho k \mu_1 \mu_2 - k\mu_1^2}{k\mu_1^2},\; k\mu_2^2 - 2\rho k \mu_1 \mu_2 + k\mu_1^2 - \frac{(\rho k \mu_1 \mu_2 - k\mu_1^2)^2}{k\mu_1^2}\right)$$

$$\sim N\left((\mu_2 - \mu_1) + \frac{(\rho\mu_2 - \mu_1)}{\mu_1}(x_1 - \mu_1),\; k\mu_2^2(1 - \rho^2)\right)$$

For $\mu_2 < \frac{\mu_1}{\rho}$, $(\rho\mu_2 - \mu_1) < 0$ and the person should invest when $x_1 < \mu_1$. For $\mu_2 = \frac{\mu_1}{\rho}$ it does not matter.

For $\mu_2 > \frac{\mu_1}{\rho}$, he should invest when $x_1 > \mu$

c)

$$Z_i \approx \ln(\mu_i) + \frac{1}{\mu_i}(X_i - \mu_i), \quad i = 1,2$$

Hence $Var(Z_i) \approx \frac{1}{\mu_i^2} Var(X_i) = \frac{k\mu_i^2}{\mu_i^2} = k, \quad i = 1,2$

$$\mu_{Z_i} \approx \ln(\mu_i), \quad i = 1,2$$

$$E[(Z_1 - \mu_{Z_1})(Z_2 - \mu_{Z_2})] \approx E\left[\frac{1}{\mu_1}(X_1 - \mu_1) \cdot \frac{1}{\mu_2}(X_2 - \mu_2)\right] = \frac{1}{\mu_1 \mu_2} Cov(X_1, X_2)$$

$= \frac{\rho k \mu_1 \mu_2}{\mu_1 \mu_2} = \rho k$. Hence $\rho_{Z_1,Z_2} \approx \frac{\rho k}{\sqrt{k}\sqrt{k}} = \rho$

$$\Sigma_{Z_2 - Z_1, Z_1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} k \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = k \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \rho - 1 & 1 \\ 1 - \rho & \rho \end{bmatrix} = k \begin{bmatrix} 2(1-\rho) & \rho - 1 \\ \rho - 1 & 1 \end{bmatrix}$$

Problem 2

) $\underline{H} = \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t = H^t$. $\quad H^2 = \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t$

$= \underline{X}(\underline{X}^t\underline{X})^{-1}X^t = H$

$J^{\circ t} = \underline{J}^{\circ}$, $\quad \underline{J}^{\circ}\underline{J}^{\circ} = \begin{bmatrix} \frac{1}{m^2}\underline{J}_m \underline{J}_m & & \\ & \ddots & \\ & & \frac{1}{m^2}\underline{J}_m \underline{J}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\underline{J}_m & & \\ & \ddots & \\ & & \frac{1}{m}\underline{J}_m \end{bmatrix} = \underline{J}^{\circ}$

$(\underline{I} - \underline{H})^2 = (\underline{I} - \underline{H})(\underline{I} - \underline{H}) = \underline{I} - \underline{H} - \underline{H} + \underline{H}^2 = I - H$. Hence $(I - H)$ is idempotent. $(I - H)^t = I - H^t = I - H \Rightarrow$ symmetric.

$(\underline{I} - \underline{J}^{\circ}) = (\underline{I} - \underline{J}^{\circ})(\underline{I} - \underline{J}^{\circ}) = \underline{I} - \underline{J}^{\circ} - \underline{J}^{\circ} + \underline{J}^{\circ}\underline{J}^{\circ} = (I - J^{\circ}) \Rightarrow$ idempotent.

$(\underline{I} - \underline{J}^{\circ})^t = I - \underline{J}^{\circ t} = \underline{I} - \underline{J}^{\circ} \Rightarrow$ symmetric.

Since $\frac{1}{n}\underline{J}$ also is idempotent and symmetric we have

rank $(\underline{I} - \underline{H}) = tr(\underline{I} - \underline{H}) = tr(\underline{I}) - tr(\underline{H}) = n - (h+1)$

rank $(\underline{I} - \underline{J}^{\circ}) = tr(\underline{I} - \underline{J}^{\circ}) = tr(\underline{I}) - tr(\underline{J}^{\circ}) = n - k$

rank $(\underline{H} - \frac{1}{n}\underline{J}) = tr(\underline{H}) - \frac{1}{n} tr(\underline{J}) = h + 1 - 1 = h$

rank $(\underline{J}^{\circ} - \frac{1}{n}\underline{J}) = tr(\underline{J}^{\circ}) - \frac{1}{n} tr(\underline{J}) = k - 1$

b)

Model 1

$H_0: \beta_1 = \beta_2 = \cdots = \beta_r = 0$ $\quad$ H_1: at least one $\beta_i \neq 0$

Model 2

$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ $\quad$ H_1: at least one $\alpha_i \neq 0$

Model 1: $$F = \frac{\frac{SS_R}{r}}{\frac{SS_E}{n-r-1}} = \frac{\frac{\underline{Y}^t(\underline{H} - \frac{1}{n}\underline{J})\underline{Y}}{r}}{\frac{\underline{Y}^t(\underline{I} - \underline{H})\underline{Y}}{n-r-1}} \sim F_{r,\, n-r-1}$$

Model 2: $$F = \frac{\frac{SS_A}{k-1}}{\frac{SS_E}{n-k}} = \frac{\frac{\underline{Y}^t(\underline{J}^\circ - \frac{1}{n}\underline{J})\underline{Y}}{k-1}}{\frac{\underline{Y}^t(\underline{I} - \underline{J}^\circ)\underline{Y}}{n-k}} \sim F_{k-1,\, n-k}$$

c) $(\underline{I} - \underline{J}^\circ)E(\underline{Y}) = \underline{\mu} + \underline{\alpha} - (\underline{\mu} + \underline{\alpha}) = \underline{0}$

Hence $\frac{\underline{Y}^t(\underline{I} - \underline{J}^\circ)\underline{Y}}{\sigma^2} = \frac{(\underline{Y} - E(\underline{Y}))^t(\underline{I} - \underline{J})(\underline{Y} - E(\underline{Y}))}{\sigma^2} \sim \chi^2(\text{rank}(\underline{I} - \underline{J}^\circ)) \sim \chi^2(n-k)$

$(\underline{J}^\circ - \frac{1}{n}\underline{J})E(\underline{Y}) = \underline{J}^\circ(\underline{\mu} + \underline{\alpha}) - \frac{1}{n}\underline{J}(\underline{\mu} + \underline{\alpha}) = \underline{\mu} - \underline{\mu}$ under H_0, since then $\underline{\alpha} = \underline{0}$

Under H_0 we then get: $\frac{\underline{Y}^t(\underline{J}^\circ - \frac{1}{n}\underline{J})\underline{Y}}{\sigma^2} = \frac{(\underline{Y} - \underline{\mu})^t(\underline{J}^\circ - \frac{1}{n}\underline{J})(\underline{Y} - \underline{\mu})}{\sigma^2} \sim \chi^2(\text{rank}(\underline{J}^\circ - \frac{1}{n}\underline{J}))$

$\sim \chi^2(k-1)$. Further, $(\underline{I} - \underline{J}^\circ)(\underline{J}^\circ - \frac{1}{n}\underline{J}) = \underline{J}^\circ - \underline{J}^\circ\underline{J}^\circ - \frac{1}{n}\underline{J} + \frac{1}{n}\underline{J}^\circ\underline{J}$

$= \underline{J}^\circ - \underline{J}^\circ - \frac{1}{n}\underline{J} + \frac{1}{n}\underline{J} = \underline{0}$. Hence the quadratic forms are independent

Thereby: $$\frac{\frac{\underline{Y}^t(\underline{J}^\circ - \frac{1}{n}\underline{J})\underline{Y}}{n-1}}{\frac{\underline{Y}^t(\underline{I} - \underline{J}^\circ)\underline{Y}}{n-k}} \sim F_{k-1,\, n-k}$$

Problem 3

a) $P(F_{2,17} \geq 29.49) = 2.97 \; 10^{-6} \Rightarrow$ that the regression is significant on a 1% level.

$SS_E = (3.138)^2 \cdot 17 = 167.4$

$$\frac{\frac{SS_R}{2}}{\frac{SS_E}{17}} = 29.49 \Rightarrow \quad SS_R = 2 \cdot (3.138)^2 \cdot 29.49 = 580.80$$

For package design 4 : $\hat{y} = 18.7 - 6.6 \cdot 4 + 2.2 \cdot 4^2 = 27.5$

b) The p-value is here $1.214 \cdot 10^{-5}$ which is less than all reasonable significance levels. Thereby $H_0: \alpha_1 = \alpha_2 = \dots = \alpha_4 = 0$ is rejected and salesnumber is dependent on package design.

$E(\bar{Y}_{4.}) = \frac{1}{5} \sum_{j=1}^{5} (\mu + \alpha_4) = \mu + \alpha_4 = \mu_4$. Hence $\bar{Y}_{4.}$ is unbiased.

For prediction intervals we need the variance of

$Y - \hat{Y}$ when Y is the new observation independent of $\hat{Y}$.

For model 1: $Var(Y - \hat{Y}) = \sigma^2 (1 + \overbrace{\underline{x}_0^T (\underline{X}^T \underline{X})^{-1} \underline{x}_0}^{h_{00}})$

For model 2 : $Var(Y - \hat{Y}) = \sigma^2 (1 + \frac{1}{5})$

Hence a $(1-\alpha)(100\%)$ prediction interval with model 1

$$27.5 \pm t_{\frac{\alpha}{2},17} \cdot 3.138 \sqrt{1 + h_{20,20}} = 27.5 \pm t_{\frac{\alpha}{2},17} \cdot 3.138 \sqrt{1 + 0.19}$$

With model 2 :

$$27.2 \pm t_{\frac{\alpha}{2},16} \cdot 3.146 \sqrt{1 + \frac{1}{5}} = 27.2 \pm t_{\frac{\alpha}{2},16} \cdot 3.146 \sqrt{1 + 0.2}$$

$t_{\frac{\alpha}{2},17} < t_{\frac{\alpha}{2},16}$, $3.138 < 3.146$, $0.19 < 0.2 \Rightarrow$ model 1 gives the shortest length of the prediction interval in this case