Testing linear hypotheses [F: 3.3]

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(1) Regression model

$$\underset{n\times 1}{Y} = \underset{n\times p}{X}\underset{p\times 1}{\beta} + \underset{n\times 1}{\varepsilon}, \quad \varepsilon \sim N_n(0, \sigma^2 I)$$

unrestriced model (A)

$\hat{\beta}, \hat{\sigma}^2$

$\hat{\varepsilon} = Y - \hat{Y} = Y - X\hat{\beta}$

$SSE = \hat{\varepsilon}^T \hat{\varepsilon}$

$\frac{SSE}{\sigma^2} \sim \chi^2_{n-p}$

Ex: Happiness

x_1, x_2, x_3, x_4

SSE

Want to test

$H_0: \beta_1 = \beta_2 = 0$ vs

$H_1:$ at least one $\neq 0$

$H_0: \underset{r\times p}{C}\beta = d$

$\beta_0 \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_4$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$H_1: C\beta \neq d$

$x_3 + x_4$ as covariates $\rightarrow$

(2) Restricted regression model (B)

Here "$C\beta = d$"

The restricted model is a special case of the unrestricted model (nested within)

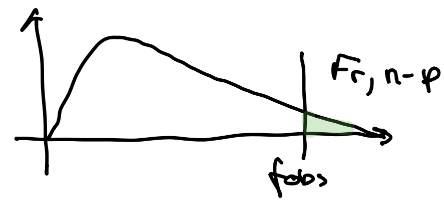
$SSE_{H_0} = \hat{\varepsilon}_{H_0}^T \hat{\varepsilon}_{H_0}$

$\hat{\beta}_{H_0}, \quad \hat{\varepsilon}_{H_0} = Y - X\hat{\beta}_{H_0}$

3

$$F_{obs} = \frac{\frac{1}{r}\left(SSE_{H_0} - SSE\right)}{\frac{1}{n-p}\, SSE} \sim F_{r,n-p}$$

when H_0 is true



understanding: when is $\underbrace{F_{obs} \text{ large}}$ → when SSE_{H_0} is much larger than SSE

↑ reason to believe that we have "missed" something when fitting restricted model.
Therefore we want to reject H_0 for large F_{obs}

Now: - why is $F_{obs} \sim F_{r,n-p}$ under H_0

- is it possible to write F_{obs} using $X, C, d, \hat{\beta}, \hat{\sigma}^2$?

We start with a new version of f_{obs}:

$$F_{obs} \overset{\circledast}{=} \frac{1}{r}\left(C\hat{\beta} - d\right)^T \left[\hat{\sigma}^2 C (X^T X)^{-1} C^T\right]^{-1} \left(C\hat{\beta} - d\right)$$

Why is this a $F_{r,n-p}$ - distribution?

1) $\hat{\beta} \sim N_p(\beta, (X^T X)^{-1}\sigma^2)$

$$Z = \underset{r\times p}{C}\underset{p\times 1}{\hat{\beta}} \sim N_r(C\beta, C\,Cov(\hat{\beta})C^T)$$

$$= N_r(\underbrace{C\beta}_{\substack{\| \\ d \text{ when } H_0 \text{ true}}}, \sigma^2 C(X^T X)^{-1}C^T)$$

2) $(C\hat{\beta}-d)^T[\sigma^2 C(X^T X)^{-1}C^T]^{-1}(C\hat{\beta}-d)$

$\sim \chi^2_r$ (Part 1)

3) σ^2 is unknown, but $\hat{\sigma}^2 = \frac{SSE}{n-p}$

and we know that $\frac{SSE}{\sigma^2} = \frac{(n-p)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-p}$

(Part 2)

4) And $\hat{\beta}$ and SSE are independent – known from Part 2.

5) So, finally:

$$\frac{\chi^2_r/r}{\chi^2_{n-p}/n-p} \sim F_{r,n-p} \quad \text{definition}$$

$$F_{obs} = \frac{\frac{1}{r}(C\hat{\beta}-d)^T[\cancel{\sigma^2}C(X^TX)^{-1}C^T]^{-1}(C\hat{\beta}-d)}{\frac{1}{\cancel{n-p}}\ \frac{\cancel{(n-p)}\hat{\sigma}^2}{\cancel{\sigma^2}}}$$

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

$$= \frac{1}{r}(C\hat{\beta}-d)^T[\hat{\sigma}^2 C(X^TX)^{-1}C^T]^{-1}(C\hat{\beta}-d)$$

Why is $F_{obs} =$?

1) For the unrestricted model $\hat{\beta} = (X^TX)^{-1}X^TY$ found by minimizing $LS(\beta) = (Y - X\beta)^T(Y - X\beta)$.

2) For the restricted model we minimize $LS(\beta)$ subject to "$C\beta = d$", by minimizing

$$LS(\beta) + 2\lambda^T(C\beta - d) = LSR(\beta)$$

$$\hat{\beta}^R = \hat{\beta} - (X^TX)^{-1}C^T[C(X^TX)^{-1}C^T](C\hat{\beta}-d)$$

see F: p 172-173

3) $\Delta SSE = \hat{\varepsilon}_{vto}^{T} \hat{\varepsilon}_{rto} - \hat{\varepsilon}^{T} \hat{\varepsilon}$

$= (Y - X\hat{\beta}^{R})^{T}(Y - X\hat{\beta}^{R}) - (Y - X\hat{\beta})^{T}(Y - X\hat{\beta})$

$= \ldots = (C\hat{\beta} - d)^{T}[C(X^{T}X)^{-1}C^{T}]^{-1}(C\hat{\beta} - d)$

F: p 173-174

4) $SSE = \hat{\sigma}^{2}(n-p)$

5) $\dfrac{\frac{1}{r}\,\Delta SSE}{\frac{1}{n-p}\,SSE} = \ldots =$

$$\boxed{\frac{1}{r}(C\hat{\beta} - d)^{T}[\hat{\sigma}^{2}C(X^{T}X)^{-1}C^{T}]^{-1}(C\hat{\beta} - d)}$$

How can we use Fobs for hypothesis testing?

Problem: $Y = X\beta + \varepsilon$, $\varepsilon \sim N_n(0, \sigma^2 I)$

$$\underbrace{H_0 : C\beta = d}_{\text{restricted model (B)}} \quad \text{vs} \quad \underbrace{H_1 : C\beta \neq d}_{\text{unrestricted model (A)}}$$

First: is the regression significant? Then maybe compare full and reduced model (Comp. Ex 3.P1) or test each $\beta_j = 0$?

Solution 1 :

Happiness

a) Fit unrestricted model and get SSE. (Ex: $X_1 + X_2 + X_3 + X_4$)

b) Fit restricted model and get SSE_{H_0} (Ex: $X_3 + X_4$)

c) $$F_{obs} = \frac{\frac{1}{r}\,\Delta SSE}{\frac{SSE}{n-p}}$$

d) Calculate p-value, $P(F_{r,n-p} > f_{obs})$, and reject or not H_0.

Solution 2

a) Fit unrestricted model $\to \hat{\beta}, \hat{\sigma}^2$

b) What is C and d

c) $F_{obs} = \circledast$ calculate this

d) Calculate the p-value.

$\Rightarrow$ Hands-on: CompEx 3. Problem 1.

Q: We had $H_0: \beta_j = 0$ vs $H_1: \beta_j \neq 0$ and used a t-test

$$T_j = \frac{\hat{\beta}_j - 0}{\widehat{SD}(\hat{\beta}_j)} \sim t_{n-p}$$

and not an F-test. Is it still the same test as using F_{obs}?

$\hat{\sigma}\sqrt{\xi_{jj}}$

A: $H_0: \beta_j = 0$ vs $H_1: \beta_j \neq 0$

where $\underset{1\times p}{C} = [0 \cdots 0 \; 1 \; 0 \cdots \cdot 0]$ and $d = 0$, $r = 1$ (the 1 at position j)

$p = \#$parameters $=$ intercept $+ (\underset{k}{p} - 1)$ covariates

$n = \#$observation

$$C\hat{\beta} = \hat{\beta}_j$$

$$[\underbrace{C\,(X^TX)^{-1}C^T}_{\underbrace{(X^TX)^{-1}_{[j,j]}}_{c_{jj}}}]^{-1} = \frac{1}{c_{jj}}$$

$$\underbrace{\frac{1}{r}}_{1}\,(\underbrace{C\hat{\beta}}_{\hat{\beta}_j}-d)^T\,[\underbrace{\hat{\sigma}^2\,C(X^TX)^{-1}C^T}]^{-1}(\underbrace{C\hat{\beta}-d}_{\hat{\beta}_j-0})$$

$$\hat{\beta}_j - 0 \qquad \frac{1}{\hat{\sigma}^2 c_{jj}}$$

$$= \frac{(\hat{\beta}_j - 0)^2}{\hat{\sigma}^2\, c_{jj}} = \underset{\substack{\uparrow \\ (t_{n-p})^2}}{T_j^2} \in F_{1,\,n-p}$$

From part 1: $(T_j)^2 = \left(\frac{Z}{\sqrt{\frac{\chi^2_\nu}{\nu}}}\right)^2 = \frac{Z^2}{\frac{\chi^2}{\nu}} \sim F_{1,\nu}$ $\qquad \frac{\chi^2_1}{1}$

$\Rightarrow$ all on, this is an F-test!