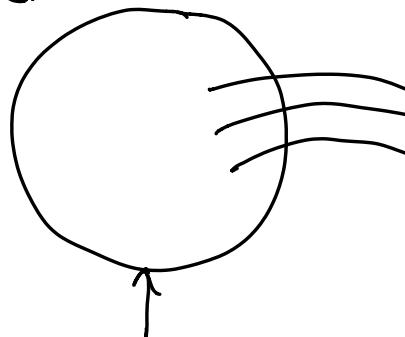


# Multiple hypothesis testing (note available from Bb)

L1b, TMA4267  
14.03.2017

## First: single hypothesis testing

Ex:



$$H_0: \mu = 120 \text{ mm Hg} \text{ vs } H_1: \mu > 120 \text{ mm Hg}$$

$$\begin{matrix} \vdots \\ X_1 \\ \vdots \\ n=100 \\ \vdots \\ X_n \end{matrix}$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$X \sim N(\mu, \sigma^2 = 10^2)$$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n} = \frac{10^2}{100} = 1)$$

$$\bar{X} \sim N(120, 1) \text{ when } H_0 \text{ true}$$

Observed  $\bar{X} = 122$ .

$$\begin{aligned} \text{P-value: } P(\bar{X} \geq 122) &= P\left(\frac{\bar{X}-120}{1} \geq \frac{122-120}{1}\right) \\ &= 1 - \Phi(2) = 0.02 \end{aligned}$$

Informally: the p-value is the probability that our test statistic ( $\bar{X}$ ) is observed to be  $\bar{X} = "122"$  or a more extreme value<sup>a</sup> (that is  $\bar{X} \geq 122$ ), when the truth is that  $\mu = 120$  so that  $\bar{X} \sim N(120, 1)$ .

Then we choose if we have enough evidence against  $H_0$  by looking at the p-value.

If the pvalue is small then what we have observed (or more extreme obs.) is not very probable when  $H_0$  is true.  $\Rightarrow$  so for small pvalues we believe that  $H_0$  must be false and reject  $H_0$ .

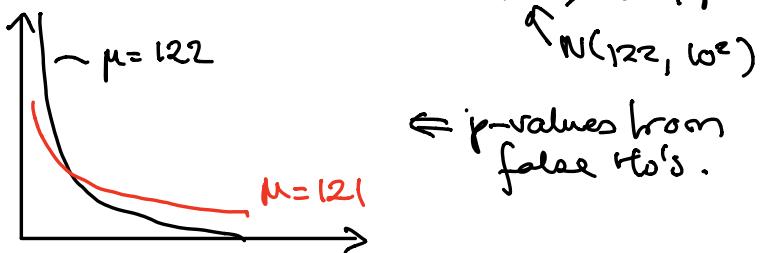
Smaller than chosen significance level  $\alpha$   
10%, 5%, 1%

$\Rightarrow$  So, the p-value can be seen as a probability?

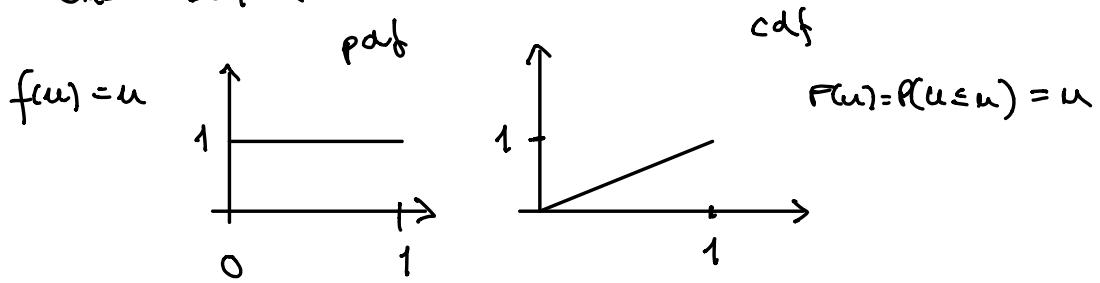
Q: What happens if I collect data on  $n=100$  new persons from the population. We observe a new  $\bar{X}$ , and will get a new pvalue.

$\Rightarrow$  The pvalue is a random variable - and it has a probability distribution.

Ex: blood pressure. Easy to sample 100 from  $N(\mu, \sigma^2 = 100)$ , calculate  $\bar{X}$  and p-value  $\Rightarrow$  make histogram.



When  $H_0$  is true the p-values are uniformly distributed.



$\Rightarrow$  see note for R-code & proof:

This is (usually) non-intuitive to people... but rather useful to know...

See note: Define valid and exact p-value -

## Multiple hypotheses

we want to decide on value for  $\alpha_{loc}$

$m = \# \text{ hypotheses}$

$R = \# \text{ hypotheses that we reject, where } p\text{value} < \alpha_{loc}$

	Not reject $H_0$	Reject $H_0$	Total
$H_0 \text{ true}$	Correct	Type I errors false positives	$m_0$
$H_0 \text{ false}$	Type II errors	Correct	$m - m_0$
Total		$R$	$m$

false news

$\nwarrow$  we only know  $m$  and  $R$

## Generalization of type I error

$$FWER = P(V > 0) = P(V \geq 1)$$

$\uparrow$   
familywise  
error rate

$\nearrow$   
one or more false news  
(false positive)

We want to control FWER — that means

to find  $\underline{\alpha_{loc}}$  so that  $P(V > 0) \leq 0.1 \leftarrow \alpha_{0.05}$

Let  $R_i = \{\text{reject } H_0 \text{ nr } i, \text{ i.e. } P_i \leq \alpha_{loc}\}$

$\bar{R}_i = \{\text{not reject } H_0 \text{ nr } i, P_i > \alpha_{loc}\}$

assume all  $H_0$  true

$$P(V > 0) = 1 - P(V = 0) \stackrel{\downarrow}{=} 1 - P(\underbrace{\bar{R}_1 \cap \bar{R}_2 \cap \dots \cap \bar{R}_m}_{\text{need the joint distribution of the } m \text{ Test statistics } T_1, \dots, T_m})$$

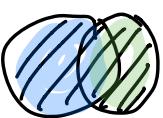
need the joint distribution of the  $m$

Test statistics  $T_1, \dots, T_m$

→ perform a multiple integral. Difficult to solve. See note on details.

Bonferroni's method: Assume all  $H_0$  are true.

$$\begin{aligned} P(V > 0) &= P(R_1 \cup R_2 \cup R_3 \cup \dots \cup R_m) \\ &\leq P(R_1) + P(R_2) + \dots + P(R_m) \end{aligned}$$

$$\left[ P(A \cup B) \leq P(A) + P(B) \quad \text{Boole's inequality} \right]$$


$$P(V > 0) \leq \underbrace{P(\text{rejecting } H_0 \text{ nr } 1)}_{P(P_i \leq \alpha_{loc})} + \dots + \underbrace{P(\text{rejecting } H_0 \text{ nr } m)}_{P(P_m \leq \alpha_{loc})}$$

$$P(\text{V} > 0) \leq \alpha_{L_0} + \alpha_{L_1} + \dots + \alpha_{L_m} = m \cdot \alpha_{L_0}$$

||

↑

PWER  
" "  
 $\alpha$

{ since p-values from true H<sub>0</sub>'s are uniform.  $P(P_i \leq \alpha_{L_0}) = \alpha_{L_0}$

Then I need to choose  $\alpha_{L_0} = \frac{\alpha}{m}$

Rule: reject H<sub>0</sub> when p-value <  $\frac{\alpha}{m}$

PWER  
α  
≠ test.

if p-values are valid  $P(P_i \leq \alpha_{L_0}) \leq \alpha_{L_0}$ , so that is also ok here.