PART 4: DOE — TMA4267 L18

Effects & Inference — 24.03.2017

* = corrected on page 5

Ex: Lime beens 2^3.

Write down the regression model with all possible interections.

$$Y_i = \beta_0 + \underset{A}{\beta_1} \cdot x_{i1} + \underset{B}{\beta_2} \cdot x_{i2} + \underset{C}{\beta_3} \cdot x_{i3}$$

$$+ \underset{AB}{\beta_{12}} x_{i1} \cdot x_{i2} + \underset{AC}{\beta_{13}} x_{i1} \cdot x_{i3} + \underset{BC}{\beta_{23}} x_{i2} \cdot x_{i3}$$

$$+ \beta_{123} x_{i1} \cdot x_{i2} \cdot x_{i3} + \varepsilon_i$$

$$\hat{\beta}_1 = \frac{1}{8} \sum_{i=1}^{8} x_{i1} \cdot Y_i = \frac{1}{8} \left(-1 \cdot 6 + 1 \cdot 4 - 1 \cdot 10 + \dots + 1 \cdot 5 \right)$$
$$= -1.125$$

$$\hat{\beta}_1 = \frac{1}{2} \underbrace{\frac{y_2 + y_4 + y_6 + y_8}{4}}_{\text{average of response when A is high}} - \frac{1}{2} \underbrace{\frac{y_1 + y_3 + y_5 + y_7}{4}}_{\text{average of response when A is low}}$$

Interpret $\hat{\beta}_1$: increase x_1 with one unit $\Rightarrow$ $\hat{y}$ increase with $\hat{\beta}_1$.

DOE Effects

For each β_j in the model (except β_0) we define an effect to be

$$\text{Effect}_j = 2 \cdot \beta_j$$

Why? β_j gives the change (in y) when x_{ij} goes from 0 to 1, while Effect_j gives the change when x_{ij} goes from -1 to 1.

Thus: $$\boxed{\hat{\text{E}}\text{ffect}_j = 2 \cdot \hat{\beta}_j}$$

This (unfortunately) means that

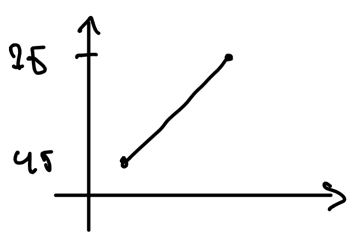
$$\text{Var}(\hat{\text{Effect}}_j) = \text{Var}(2 \cdot \hat{\beta}_j) = 4 \cdot \underbrace{\text{Var}(\hat{\beta}_j)}_{= \frac{\sigma^2}{n}} = \underline{\underline{\frac{4\sigma^2}{n}}}$$

insert $\hat{\sigma}^2 \Rightarrow s^2_{\text{effect}_j}$

WARNING:

DOE main effect: $2\hat{\beta}_j$ for A, B, C

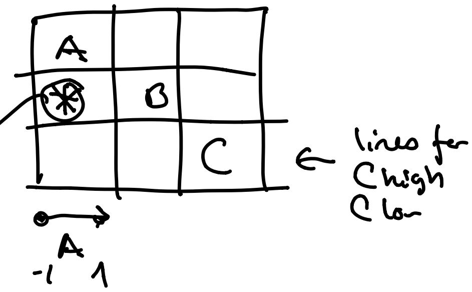
shown in main effects plot



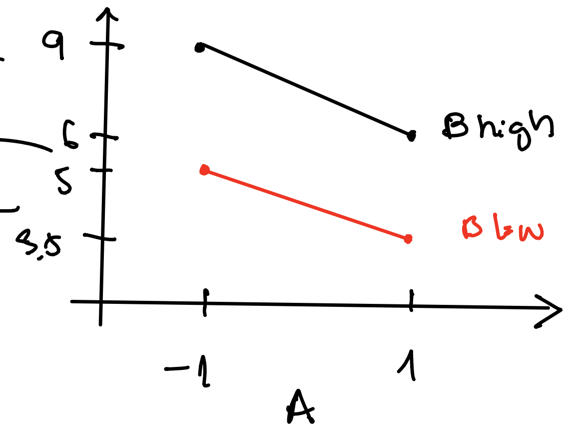
$\hat{B} = 3.25$

Interaction effect.

Ex: $\hat{AB} = 2\hat{\beta}_{12}$



	A low	A high
B low	5	3.5
B high	9	6



$\hat{AB} = -0.75$

$$\hat{AB} = \tfrac{1}{2}(\text{est. main effect of } A \text{ when } B \text{ is high})$$
$$- \tfrac{1}{2}(\text{— " — low})$$
$$= \tfrac{1}{2}(6-9) - \tfrac{1}{2}(3.5-5) = -0.75$$

rather small effect

When the two lines in the interaction plot are parallel ⇒ there is <u>no</u> interaction effect.

⇒ DOE report (Ex 4) ← add explanation for one interaction effect.

Significant effects

$$H_0: \underset{2\beta_j}{\text{Effect}_j} = 0 \quad \text{vs} \quad H_1: \text{Effect}_j \neq 0$$

(or equivalently: $H_0: \beta_j = 0$ vs $H_1: \beta_j \neq 0$)

$$\text{Effect}_j = 2\beta_j$$

$$\widehat{\text{Effect}}_j = 2\hat{\beta}_j = \frac{2}{n}\sum_{i=1}^{n} x_{ij} \cdot y_i$$

$$E(\widehat{\text{Effect}}_j) = 2 \cdot E(\hat{\beta}_j) = 2\beta_j = \text{Effect}_j$$

$$\text{Var}(\widehat{\text{Effect}}_j) = 4 \cdot \text{Var}(\hat{\beta}_j) = 4 \cdot \frac{1}{n}\sigma^2 \equiv \sigma^2_{\text{effect}}$$

NBNB not dependent on j

$$\widehat{\text{Effect}}_j \sim N(\text{Effect}_j, \sigma^2_{\text{effect}})$$

If we have S^2effect as an estimator for σ^2effect.
we might get

$$T_j = \frac{\widehat{\text{Effect}}_j - \text{Effect}_j}{S_{\text{effect}}} \sim t_\nu$$

ν is dependent on S^2effect.

95% CI: $\left[\widehat{Effect}_j \pm t_{\frac{\alpha}{2}, \nu} \cdot S_{effect} \right]$

Hypothesis test: reject H_0 when

$$|t_j| = \left| \frac{\widehat{Effect}_j - 0}{S_{effect}} \right| > t_{\frac{\alpha}{2}, \nu}$$

numerical value

$$|\widehat{Effect}_j| > t_{\frac{\alpha}{2}, \nu} \cdot S_{effect}$$

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1) Perform replication of a full 2^4 design. → use lm as before.

Pareto-plot: barplot (horizontal) of $\widehat{Effect}_j$ with red line at $\underline{t_{\frac{\alpha}{2}, \nu} \cdot S_{effect}}$

Ex
Lima beans: 3 replicates of 8 observations $\underline{n = 24}$

Estimating 8 parameters (μ_0 + A, B, C, AB, AC, BC, ABC)

$\Rightarrow n - p = 24 - 8 = 16 \leftarrow \underline{\nu = 16}$

Residual standard error in printout

$$S_{effect} = \sqrt{\frac{4}{n} \cdot \hat{\sigma}^2} = \sqrt{\frac{4}{24}} \cdot S = 0.3$$

$S = 0.736$

$$t_{\frac{0.05}{2}, 16} = 2.12$$

0.64 = red line

$2.12 \cdot 0.3 = t_{\frac{\alpha}{2}, \nu} \cdot S_{effect}$

2) Fit reduced model → read off:

Curiosity: S_{effect} can be calculated from the full model by $S_{effect}^2 = \frac{1}{m} \sum_{} Effect_j^2$

alle $Effect_j$ not part of model

3) Lenth's method.