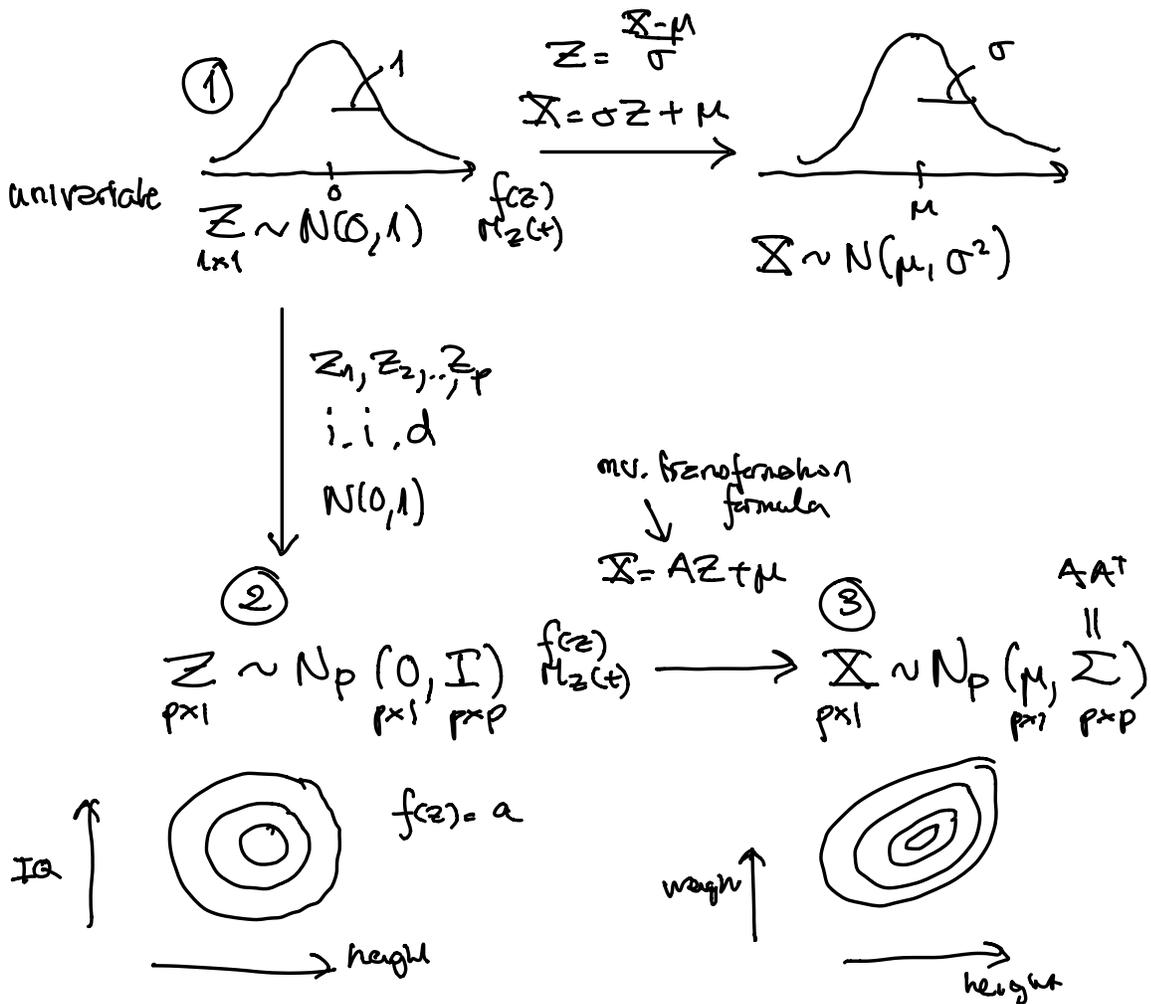


Multivariate normal distribution

Plan: derive the pdf and moment generating function (MGF $M_Z(t) = E(e^{t^T Z})$) of the multivariate normal.



Characterizing \mathbf{X} :

- $f(x)$ (pdf), $F(x)$ (cdf)

$$- M_{\mathbf{X}}(\mathbf{t}) = E\left(e^{\mathbf{t}^T \mathbf{X}}\right) = E\left(e^{t_1 X_1 + t_2 X_2 + \dots + t_p X_p}\right)$$

$p \times 1$ $p \times 1$

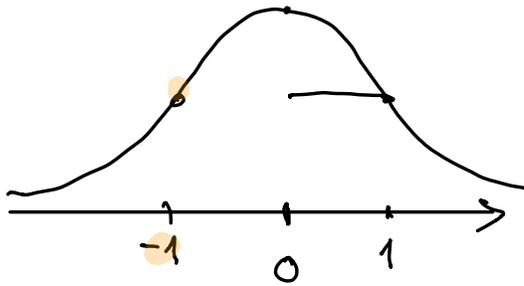
$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{\mathbf{t}^T \mathbf{x}} f(\mathbf{x}) dx_1 \dots dx_p$$

MGF : easy for proofs.

① $Z \sim N(0, 1)$

$E(Z) = 0, \text{Var}(Z) = 1$

$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad -\infty < z < \infty$



$f'(z) = 0$ for $z = 0$
 $f''(z) = 0$ for $z = 1$ and $z = -1$
 $z \rightarrow \pm\infty \Rightarrow f(z) \rightarrow 0$

↑ mean, median, mode

MGF (univ) $M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tZ} f(z) dz$

$= \dots = e^{\frac{1}{2}t^2}$
 (homework) ↑ percent.

Cool: The law of the unconscious statistician (brevistles)

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$

↑ $E(Y) = \int_{-\infty}^{\infty} y f(y) dy$

① → ②

Z_1, Z_2, \dots, Z_p independent $N(0,1)$ RVs

$$f(\mathbf{z}) = \prod_{i=1}^p \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_i^2} = \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^p z_i^2\right\}$$

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{pmatrix} \quad = \underline{\underline{\left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \exp\left\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right\}}}$$

$$E(\mathbf{z}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{p \times 1} \quad \text{and} \quad \text{Cov}(\mathbf{z}) = \mathbf{I}_{p \times p}$$

$$\text{Var}(Z_i) = 1$$

$$\text{Cov}(Z_i, Z_j) = 0 \quad i \neq j$$

MGF: multivariate version

$$M_{\mathbf{z}}(\mathbf{t}) \stackrel{\text{DEF}}{=} E\left(\exp(\mathbf{t}^T \mathbf{z})\right)$$

$$t_1 z_1 + t_2 z_2 + \dots + t_p z_p$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(t_1 z_1 + t_2 z_2 + \dots + t_p z_p) \cdot \underbrace{f(\mathbf{z})}_{\prod_{i=1}^p f(z_i)} dz_1 \dots dz_p$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{t_1 z_1} \cdot e^{t_2 z_2} \dots e^{t_p z_p} \cdot f(z_1) \cdot f(z_2) \dots f(z_p) dz_1 \dots dz_p$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{t_1 z_1} f(z_1) dz_1}_{M_{z_1}(t)} \dots \underbrace{\int_{-\infty}^{\infty} e^{t_p z_p} f(z_p) dz_p}_{M_{z_p}(t)}$$

$$M_z(t) = \prod_{i=1}^p \underbrace{M_{z_i}(t)}_{e^{\frac{1}{2} t_i^2}} = \prod_{i=1}^p e^{\frac{1}{2} t_i^2} = \underline{\underline{e^{\frac{1}{2} t^T t}}}$$

$e^{\frac{1}{2} t_1^2} \cdot e^{\frac{1}{2} t_2^2} \dots e^{\frac{1}{2} t_p^2}$

② → ③

↑

$$\underset{p \times 1}{Z} \sim N_p(0, I)$$

$$f(z) = \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} e^{-\frac{1}{2}z^T z}$$

$$M_Z(t) = e^{\frac{1}{2}t^T t}$$

Now: $\underset{p \times 1}{X} = \underset{p \times p}{A} \underset{p \times 1}{Z} + \underset{p \times 1}{\mu}$, where A has full rank
 $Ax = 0$ has only solution $x=0$

$$E(X) = A E(Z) + E(\mu) = A \cdot 0 + \mu = \mu$$

$$\text{Cov}(X) = A \underbrace{\text{Cov}(Z)}_I A^T = A A^T \equiv \Sigma$$

since A has full rank

$A A^T$ is positive definite

(can be proven using

$$(Ax)^T (Ax)$$

What can A be? $\Sigma^{\frac{1}{2}}$, but also other possibilities.

② → ③ First MGF of \mathbb{X}

$$M_{\mathbb{X}}(t) = M_{A\mathbb{Z} + \mu}(t) = E\left(e^{t^T(A\mathbb{Z} + \mu)}\right)$$
$$= E\left(e^{t^T A \mathbb{Z}} \cdot e^{t^T \mu}\right) = e^{t^T \mu} E\left(e^{(t^*)^T A \mathbb{Z}}\right)$$

$$t^* = A^T t$$

$$= e^{t^T \mu} M_{\mathbb{Z}}(A^T t) \quad M_{\mathbb{Z}}(t) = E(e^{t^T \mathbb{Z}})$$

$$= e^{t^T \mu} e^{\frac{1}{2} t^T A A^T t} = \underline{\underline{e^{t^T \mu + \frac{1}{2} t^T \Sigma t}}}}$$

② → ③ $f(x)$, and the mv. transformation formula

The mv. transf. formula

$$\mathbb{X} = A\mathbb{Z} + \mu \Leftrightarrow \boxed{\mathbb{Z} = A^{-1}(\mathbb{X} - \mu)}$$

$$f_{\mathbb{X}}(x) = f_{\mathbb{Z}}(z(x)) \cdot \text{abs}(J)$$

$$J = \det \left(\left(\frac{\partial z_i}{\partial x_j} \right)_{i,j} \right) \quad \text{element } (i,j)$$

↑ a matrix with this as J

We have $\boxed{Z = A^{-1}X - A^{-1}\mu}$

$$\left\{ \frac{\partial z_i}{\partial x_j} \right\} = A^{-1}$$

$$f(x) = \left(\frac{1}{2\pi} \right)^{\frac{p}{2}} \exp \left\{ -\frac{1}{2} \left(A^{-1}(x-\mu) \right)^T A^{-1}(x-\mu) \right\}$$

$$\text{abs}(\det(A^{-1})) = a$$

yes, this is a

$$\downarrow$$

a) $\text{abs}(\det(A^{-1})) = \dots = \underline{\det(\Sigma)^{-\frac{1}{2}}}$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(A)$$

$$\det(\Sigma) = \det(AA^T) = \det(A) \cdot \det(A^T)$$

$$= [\det(A)]^2 \quad \text{so } \underline{\det(A) = [\det(\Sigma)]^{\frac{1}{2}}}$$

and since Σ is SPD $\det(\Sigma) > 0$ so \uparrow yes, this is correct

$$\text{abs}(\det(A^{-1})) = \det(\Sigma)^{-\frac{1}{2}}$$

b) $\Sigma A^{-1} (\mathbf{X} - \mu) \mathbf{J}^T A^{-1} (\mathbf{X} - \mu) =$
 yes, this is b $(\mathbf{X} - \mu)^T \underbrace{(A^{-1})^T A^{-1}}_{\Sigma^{-1}} (\mathbf{X} - \mu) = \underline{(\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu)}$

because

$$\Sigma^{-1} = (A A^T)^{-1} = (A^T)^{-1} A^{-1} = (A^{-1})^T A^{-1}$$

$$f(\mathbf{z}) = \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} [\det(\Sigma)]^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} (\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu)\right\}$$

Observe byproduct:

$$(\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu) = \mathbf{Z}^T \mathbf{Z} = \begin{matrix} \chi_1^2 & \chi_1^2 & \dots & \chi_p^2 \\ \downarrow & \downarrow & \dots & \downarrow \\ z_1^2 & z_2^2 & \dots & z_p^2 \end{matrix}$$

$(N(0,1))^2$

$\sim \chi_p^2$

$$Z_i \sim N(0,1)$$

$$Z_i^2 \sim \chi_1^2 \leftarrow \text{Rec Ex 1. PS}$$

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Comment: when A does not have full rank
we can use a singular version of the pdf.

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