# TMA4267 Linear Statistical Models <br> Part 1: Multivariate random variables and the multivariate normal distribution <br> Recommended exercise 1 - V2017 

January 10, 2017

Keywords: random vectors, random matrices, simple matrix algebra, mean and covariance of linear combinations of random variables, symmetric positive definite matrices, square root matrix, R

- This first exercise starts with simple matrix algebra done by hand and by using the statistical software R.
- Formulas for the mean and variance-covariance of linear combinations of random variables, $\boldsymbol{C X}$, are tested in Problem 2
- The definition of the covariance between two random vectors appears in Problem 3.
- Symmetric and positive definite matrices play an important role in statistics, and the Mahalanobis distance is useful in the development of the multivariate normal distribution, and is the focus of Problem 4. In addition the spectral decomposition theorem (diagonalization) is used to define a square root matrix.
- We will use (multivariate) moment generating functions and the (multivariate) transformation formula for proving properties for the multivariate normal distribution, and start here in Problem 5 and 6 with going from a standard normal to a $\chi^{2}$ distribution (univariately) in theory and then check the results by simulation.
- An idempotent matrix $\boldsymbol{A}$ is a matrix where $\boldsymbol{A}=\boldsymbol{A}^{2}$. Projection matrices are idempotent, and will be important in Parts 1-3. Here we start in Problem 7 by looking at properties of an idempotent matrix.


## Problem 1: Simple matrix calculations

Solve the problems by hand AND by use of R (when possible).
The matrix $A$ is given by

$$
A=\left[\begin{array}{rr}
9 & -2 \\
-2 & 6
\end{array}\right]
$$

a) Construct $A$ as a matrix in R.

Command: matrix.
b) Is $A$ symmetric? Command: t.
c) Show that $A$ is positive definite. (Not in R. Use the direct definition of positive definiteness.)
d) Find the eigenvalues and the eigenvectors of the matrix. $A$. Are the eigenvectors found by R normalized?
In $R$ matrix multiplication is performed by the command $\% * \%$.
Command: eigen.
e) Write the spectral decomposition of the matrix $A$. In R matrix multiplication is performed by the command $\% * \%$.
f) Find $A^{-1}$.

Command: solve.
g) Find the eigenvalues and the eigenvectors of the matrix $A^{-1}$. Is there a relationship between the eigenvalues and the eigenvectors of $A$ and $A^{-1}$ ?
Command: solve, eigen
h) Why can $A$ be a covariance matrix?
i) Assume $A$ is a covariance matrix. Find the correlation matrix.

Command: diag, sqrt Check your computations with cov2cor.
j) Let $\mathbf{X}$ be a random vector, and let

$$
\begin{align*}
\mathrm{E}(\mathbf{X}) & =\left[\begin{array}{l}
3 \\
1
\end{array}\right],  \tag{1}\\
\operatorname{Cov}(\mathbf{X}) & =A, \tag{2}
\end{align*}
$$

and

$$
\begin{align*}
B & =\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right],  \tag{3}\\
\mathbf{d} & =\left[\begin{array}{l}
1 \\
2
\end{array}\right] . \tag{4}
\end{align*}
$$

Find in $R$ the expectation and covariance for:

$$
\begin{align*}
\mathbf{s} & =B \mathbf{X}  \tag{5}\\
t & =\mathbf{d}^{\prime} \mathbf{X}  \tag{6}\\
\mathbf{v} & =\left[\begin{array}{c}
\mathbf{X} \\
3 \mathbf{X}
\end{array}\right] \tag{7}
\end{align*}
$$

## Problem 2: Mean and covariance of linear combinations

Let $\boldsymbol{X}=\left(\begin{array}{c}X_{1} \\ X_{2} \\ X_{3}\end{array}\right)$ be a trivariate random vector with mean $\boldsymbol{\mu}=\mathrm{E}(\boldsymbol{X})=\left(\begin{array}{c}1 \\ 1 \\ 1\end{array}\right)$ and covariance matrix $\boldsymbol{\Sigma}=\operatorname{Cov}(\boldsymbol{X})=\boldsymbol{I}=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$. Further, let $\boldsymbol{A}=\left(\begin{array}{rrr}\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{array}\right)$ be a matrix of constants.
Define $\boldsymbol{Y}=\left(\begin{array}{c}Y_{1} \\ Y_{2} \\ Y_{3}\end{array}\right)=\boldsymbol{A} \boldsymbol{X}$.
Find the mean and covariance matrix of $\boldsymbol{Y}$.

## Problem 3: Covariance formula

Let $\boldsymbol{V}$ be a random vector with mean $\boldsymbol{\mu}$, and $\boldsymbol{W}$ be a random vector with mean $\boldsymbol{\eta}$. The covariance between $\boldsymbol{V}$ and $\boldsymbol{W}$ is defined as

$$
\operatorname{Cov}(\boldsymbol{V}, \boldsymbol{W})=\mathrm{E}\left((\boldsymbol{V}-\boldsymbol{\mu})(\boldsymbol{W}-\boldsymbol{\eta})^{T}\right)
$$

Show that

$$
\operatorname{Cov}(\boldsymbol{V}, \boldsymbol{W})=\mathrm{E}\left(\boldsymbol{V} \boldsymbol{W}^{T}\right)-\boldsymbol{\mu} \boldsymbol{\eta}^{T}
$$

Observe that this also implies that $\operatorname{Cov}(\boldsymbol{V})=\mathrm{E}\left(\boldsymbol{V} \boldsymbol{V}^{T}\right)-\boldsymbol{\mu} \boldsymbol{\mu}^{T}$, and can be seen as a generalization of the well known $\operatorname{Var}(V)=\mathrm{E}\left(V^{2}\right)-\mu^{2}$ for (univariate) random variable $V$ with mean $\mu$.

## Problem 4: The square root matrix and the Mahalanobis transform

Let the expectation (mean) and covariance matrix for a $p$-variate random vector $\boldsymbol{X}$ be $\boldsymbol{\mu}=\mathrm{E}(\boldsymbol{X})$ and $\boldsymbol{\Sigma}=\operatorname{Cov}(\boldsymbol{X})$.

Let further $\left(\lambda_{i}, \boldsymbol{e}_{i}\right), i=1, \ldots, p$ be the eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$. Let $\boldsymbol{P}$ be the matrix of eigenvectors,

$$
\boldsymbol{P}=\left[\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \ldots, \boldsymbol{e}_{p}\right]
$$

and $\boldsymbol{\Lambda}$ be a diagonal matrix with the eigenvalues $\lambda_{1}, \lambda_{1}, \ldots, \lambda_{p}$ on the diagonal.
a) Show that if $\boldsymbol{\Sigma}$ is symmetric and positive definite then all eigenvalues of $\boldsymbol{\Sigma}$ are positive.
(Hint: a symmetric matrix $\boldsymbol{A}$ is positive definite if $\boldsymbol{z}^{T} \boldsymbol{A} \boldsymbol{z}>0$ for all vectors $\boldsymbol{z} \neq \mathbf{0}$ ).
What can you say about the eigenvalues and eigenvectors to the inverse matrix of $\boldsymbol{\Sigma}$ ? Justify the answer.
b) Define the matrices $\boldsymbol{\Sigma}^{\frac{1}{2}}$ and $\boldsymbol{\Sigma}^{-\frac{1}{2}}$ by

$$
\begin{aligned}
\boldsymbol{\Sigma}^{\frac{1}{2}} & =\boldsymbol{P} \boldsymbol{\Lambda}^{\frac{1}{2}} \boldsymbol{P}^{T} \\
\boldsymbol{\Sigma}^{-\frac{1}{2}} & =\boldsymbol{P} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{P}^{T}
\end{aligned}
$$

Show that both matrices are symmetric and that the following is true:

$$
\begin{aligned}
\boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\Sigma}^{\frac{1}{2}} & =\boldsymbol{\Sigma} \\
\boldsymbol{\Sigma}^{-\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} & =\boldsymbol{\Sigma}^{-1} \\
\boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} & =\boldsymbol{I}
\end{aligned}
$$

where $\boldsymbol{I}$ is the identity matrix.
c) The transform

$$
\boldsymbol{Y}=\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{X}-\boldsymbol{\mu})
$$

is called the Mahalanobis transform. Show that $\mathrm{E}(\boldsymbol{Y})=0$ and $\operatorname{Cov}(\boldsymbol{Y})=\boldsymbol{I}$.

## Problem 5: The standard normal and chi-square distribution

The aim of this problem is to start working with moment generating functions and transformations.
a) Let $U \sim N(0,1)$. Find the pdf of $U^{2}$. Also find the moment generating function (MGF) of $U^{2}$.
b) Let $V \sim \chi_{p}^{2}$. Show that the pdf of $V$ equals

$$
f_{V}(v)=\frac{1}{\Gamma(p / 2) 2^{p / 2}} v^{(p / 2)-1} e^{-v / 2}
$$

Hint: first use the result from a) to find the MGF for the $\chi_{p}^{2}$. Then find the MGF of $V$ from $f_{V}(v)$. Show that the two coincide.

## Problem 6: N and chi-square by simulation - in R

Let dist denote a given distribution, e.g. norm, chisq, $t$, or $f$. In $R$ we have functions to sample from a distibution (prefix r), calculate the pdf (prefix d), calculate the cdf (prefix p), and calculate a critical value (prefix $q$ ).
Let $B=10000$ and $n=10$.
a) Find out more about the combinations of prefix and distribution names by typing ?rnorm, and some of the other combinations.
b) Start with the normal distribution. Make a plot of the standard normal pdf. Then add vertical lines at the critical values for the 0.05 and 0.95 quantiles. Color the tails, e.g. by using the polygon function.
c) Move to the $\chi^{2}$. Simulate $B$ data points from the standard normal distribution and square the data. Plot a histogram of the data. Add the pdf of the $\chi_{1}$ to the histogram. Then add vertical lines for the the critical values for the 0.1 and 0.9 quantiles.

## Problem 7: Symmetric, idempotent matrices

A symmetric matrix $\boldsymbol{A}$ is idempotent if $\boldsymbol{A}^{2}=\boldsymbol{A}$.
a) Prove that the eigenvalues of an idempotent matrix are 0 and 1 .
b) Assume that it is known that the rank of a symmetric matrix (actually: a diagonalizable quadratic matrix) equals the number of nonero eigenvalues of the matrix. Use the result in a) together with this result to show (understand) the following: If a $(n \times n)$ symmetric idempotent matrix $\boldsymbol{A}$ has rank $r$ then $r$ eigenvalues are 1 and $n-r$ are 0 .
c) What is the relationship between the trace and rank of a symmetric idempotent matrix?
d) Define the matrix

$$
\boldsymbol{J}=\mathbf{1 1}^{T}=\left[\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1
\end{array}\right]
$$

Show that the following matrices are symmetric and idempotent: $\frac{1}{n} \boldsymbol{J}$, and $\left(\boldsymbol{I}-\frac{1}{n} \boldsymbol{J}\right)$. Also find the rank (or trace) of the matrices.

