Example (2019, Problem 4)

$$Y_1 = Y_{11} = \mu_1 \cdot 1 + \mu_2 \cdot 0 + \mu_3 \cdot 0 + \varepsilon_{11} \quad (\varepsilon_1)$$

...

$$Y_{10} = Y_{10,1} = \mu_1 \cdot 1 + \mu_2 \cdot 0 + \mu_3 \cdot 0 + \varepsilon_{10,1} \quad (\varepsilon_{10})$$

$$Y_{11} = Y_{12} = \mu_1 \cdot 0 + \mu_2 \cdot 1 + \mu_3 \cdot 0 + \varepsilon_{12} \quad (\varepsilon_{11})$$

...

$$Y_{20} = Y_{10,2} = \mu_1 \cdot 0 + \mu_2 \cdot 1 + \mu_3 \cdot 0 + \varepsilon_{10,2} \quad (\varepsilon_{20})$$

$$Y_{21} = Y_{13} = \mu_1 \cdot 0 + \mu_2 \cdot 0 + \mu_3 \cdot 1 + \varepsilon_{13} \quad (\varepsilon_{21})$$

...

$$Y_{30} = Y_{10,3} = \mu_1 \cdot 0 + \mu_2 \cdot 0 + \mu_3 \cdot 1 + \varepsilon_{10,3} \quad (\varepsilon_{30})$$

$$Y = X\beta + \varepsilon \qquad \beta = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \qquad n = 30 \quad p = 3$$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ \cdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots \\ 0 & 0 & 1 \end{bmatrix} \qquad X^T X = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$(X^T X)^{-1} = \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

$$\hat{\mu}_2 = 0.2488 \qquad \hat{\mu}_3 = 1.1663 \qquad SSE = 24.00$$

a) $H_0: \mu_2 = \mu_3 \qquad H_1: \mu_2 \neq \mu_3$

$\alpha = 0.05$

[Recall: general F-test. A $(r \times p)$ matrix

d $(r \times 1)$ vector, $r \leq p$, $\text{rank}(A) = r$

$H_0: A\beta = d \qquad H_1: A\beta \neq d$

significance level α

Test statistic

$$F = \frac{(SSE^R - SSE)/r}{SSE/(n-p)} =$$

$$= \frac{(A\hat{\beta}-d)^T (A(X^TX)^{-1}A^T)^{-1}(A\hat{\beta}-d)/r}{(Y-X\hat{\beta})^T(Y-X\hat{\beta})/(n-p)}$$

Under H_0

$$F \sim F_{r,n-p}$$

Test

$$F \geq f_{\alpha, r, n-p} \Longrightarrow H_0 \text{ is rejected }]$$

In our case

$$A = [0 \ \ 1 \ \ -1] \quad r=1 \quad d=0$$

$$A\hat{\beta} - d = \mu_2 - \mu_3 = (A\hat{\beta}-d)^T$$

$$A(X^TX)^{-1}A^T = [0 \ \ 1 \ \ -1] \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$$

$$= \left[0 \ \ \frac{1}{10} \ \ -\frac{1}{10}\right] \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$$

Therefore the numerator of the test statistic is $5(\mu_2-\mu_3)^2 = 5 \cdot (1.1663 - 0.2488)^2 =$

$$= 4.21$$

The denominator is

$$SSE/(n-p) = 24/27 = 0.89$$

The observed value of the test statistic

$$F = 4.73$$

$$f_{0.05, 1, 27} = 4.21$$

Thus $H_0: \mu_2 = \mu_3$ is rejected.

b)

	$\mu_1 = \mu_2$	$\mu_1 = \mu_3$	$\mu_2 = \mu_3$
p-value	0.784	0.021	0.038

H_0 is rejected if p-value $\leq \alpha_{loc} = \frac{1}{m}$ FWER

where m is the number of hypotheses. In our case $m = 3$, $FWER = 0.05$, therefore
$\alpha_{loc} = \frac{0.05}{3} = 0.017$.

None of the null hypotheses are rejected.

c) Denote hypotheses
$H_{01}: \mu_1 = \mu_2$, $H_{02}: \mu_1 = \mu_2$, $H_{03}: \mu_2 = \mu_3$,
$H_0: \mu_1 = \mu_2 = \mu_3$

Define events
$A_1 = \{ H_{01} \text{ is rejected} \}$
$A_2 = \{ H_{02} \text{ is rejected} \}$
$A_3 = \{ H_{03} \text{ is rejected} \}$
$B = \{ H_0 \text{ is rejected} \}$

Four situations are possible. Consider them separately.

1) All three H_{01}, H_{02}, H_{03} are true. Then H_0 is also true, and
$FWER = P(\text{at least one Type I error}) =$
$= P_{H_0}((A_1 \cup A_2 \cup A_3) \cap B) \leq P_{H_0}(B) \leq 0.05$

2) Two of H_{01}, H_{02}, H_{03} are true. Then the third is also true and H_0 is also true, and we get the previous situation

3) One of H_{01}, H_{02}, H_{03} is true (for example, let it be H_{01}). Then
$FWER = P(\text{at least one Type I error}) =$
$= P(\text{Type I error for } H_{01}) =$
$= P_{H_{01}}(A_1 \cap B) \leq P_{H_{01}}(A_1) \leq 0.05$

4) No one of H_{01}, H_{02}, H_{03} is true

<u>Type I error is impossible</u> (for each H_{01}, H_{02}, H_{03})
For the given data

$H_{02}: \mu_1 = \mu_3$ and $H_{03}: \mu_2 = \mu_3$

are rejected because their p-values are less than 0.05, and <u>p-value</u> of $H_0: \mu_1 = \mu_2 = \mu_3$ is less than 0.05.