**Solutions** (TMA4267 2023 May)

**1.**
**a)** Denote

$$A = \begin{pmatrix} a & a_1 \\ a_1 & a \end{pmatrix}.$$

Then

$$EY = AEX = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & a_1 \\ a_1 & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a + a_1 \\ a_1 + a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

therefore $a_1 = 1 - a$.

The covariance matrix of $Y$ is

$$\mathrm{Cov}(Y) = A\Sigma A^T = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}\begin{pmatrix} 13/5 & 1 \\ 1 & 13/5 \end{pmatrix}\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{16}{5}a^2 - \frac{16}{5}a + \frac{13}{5} & -\frac{16}{5}a^2 + \frac{16}{5}a + 1 \\ -\frac{16}{5}a^2 + \frac{16}{5}a + 1 & \frac{16}{5}a^2 - \frac{16}{5}a + \frac{13}{5} \end{pmatrix}.$$

Since components of $Y$ are independent, this matrix must be diagonal, i.e.

$$-\frac{16}{5}a^2 + \frac{16}{5}a + 1 = 0.$$

Solving this equation, we obtain two solutions $a = \frac{5}{4}$ and $a = -\frac{1}{4}$. Thus either

$$A = \begin{pmatrix} 5/4 & -1/4 \\ -1/4 & 5/4 \end{pmatrix} \text{ or } A = \begin{pmatrix} -1/4 & 5/4 \\ 5/4 & -1/4 \end{pmatrix}.$$

It is easy to check that both matrices satisfy conditions of the problem. In fact, they give the same $\mathrm{Cov}(Y)$.

**b)** The only matrix satisfying this condition is the zero matrix

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Indeed, from the course we know that if $X$ is normal, then $AX$ and $BX$ are independent (for some matrices $A$ and $B$) iff

$$\mathrm{Cov}(AX, BX) = A\Sigma B^T = 0.$$

In our case $A = C$, $B = I$. Denote

$$C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

Then

$$0 = C\Sigma = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} 13/5 & 1 \\ 1 & 13/5 \end{pmatrix} = \begin{pmatrix} \frac{13}{5}c_1 + c_2 & c_1 + \frac{13}{5}c_2 \\ \frac{13}{5}c_3 + c_4 & c_3 + \frac{13}{5}c_4 \end{pmatrix}$$

or

$$\frac{13}{5}c_1 + c_2 = 0,\ c_1 + \frac{13}{5}c_2 = 0,$$
$$\frac{13}{5}c_3 + c_4 = 0,\ c_3 + \frac{13}{5}c_4 = 0.$$

This is possible only if

$$c_1 = c_2 = c_3 = c_4 = 0.$$

**2.**

**a)** The covariance matrix of $\hat{\beta}$ is $\sigma^2(X^TX)^{-1}$. In our case

$$X^TX = \begin{bmatrix} 30 & 20 & 20 \\ 20 & 20 & 10 \\ 20 & 10 & 20 \end{bmatrix},$$

$$(X^TX)^{-1} = \begin{bmatrix} 0.3 & -0.2 & -0.2 \\ -0.2 & 0.2 & 0.1 \\ -0.2 & 0.1 & 0.2 \end{bmatrix},$$

therefore the correlation coefficient between $\hat{\beta}_1$ and $\hat{\beta}_2$ is

$$\frac{\text{Cov}(\hat{\beta}_1, \hat{\beta}_2)}{\sqrt{\text{Var}(\hat{\beta}_1)}\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\sigma^2 \cdot 0.1}{\sqrt{\sigma \cdot 0.2}\sqrt{\sigma \cdot 0.2}} = 0.5.$$

**b)** Consider the followin matrix: $A = [0\ 1\ -2]$. The null hypothesis and alternative can be written as

$$H_0 : A\beta = 0 \text{ vs. } H_1 : A\beta \neq 0.$$

We use the general $F$-test. Test statistic is

$$F = \frac{(A\hat{\beta})^T(A(X^TX)^{-1}A^T)^{-1}A\hat{\beta}}{\hat{\sigma}^2}.$$

Under $H_0$ this statistic has $F$-distribution with 1 and $n-p$ degrees of freedom. In our case $n = 30$, $p = 3$, $\hat{\beta}_1 = 2.579$, $\hat{\beta}_2 = 1.196$, $\hat{\sigma} = 2.243$,

$$A(X^TX)^{-1}A^T = 0.6,$$

$$(A(X^TX)^{-1}A^T)^{-1} = \frac{10}{6} = 1.67,$$

$$A\hat{\beta} = \hat{\beta}_1 - 2\hat{\beta}_2 = 0.187.$$

Thus the observed value of the test statistic is $F = 0.012$. Since $f_{0.05,1,27} = 4.21$, the null hypothesis is not rejected.

**c)**

`Estimate`

is $\hat{\beta}_j$ $(j = 0, 1, 2)$;

`Std. Error`

is $\sqrt{\widehat{\text{Var}\hat{\beta}_j}}$;

`t value`

is $t_j = \hat{\beta}_j / \sqrt{\widehat{\text{Var}\hat{\beta}_j}}$.

For the first question mark

$$\hat{\beta}_0 = t_0 \sqrt{\widehat{\text{Var}\hat{\beta}_0}} = 0.776 \cdot 1.228 = 0.953.$$

For the second question mark

$$\sqrt{\widehat{\text{Var}\hat{\beta}_1}} = \hat{\beta}_1 / t_1 = 2.579/2.571 = 1.003.$$

For the third question mark

$$t_2 = \hat{\beta}_2 / \sqrt{\widehat{\text{Var}\hat{\beta}_2}} = 1.196/1.003 = 1.193.$$

Thus the output is in fact as follows

```
Call:
lm(formula = Y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-4.1806 -1.2880  0.3316  1.3483  4.7308

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.953      1.228   0.776    0.445
x1             2.579      1.003   2.571    0.016 *
x2             1.196      1.003   1.193    0.243
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.243 on 27 degrees of freedom
Multiple R-squared:  0.197,     Adjusted R-squared:  0.1375
F-statistic: 3.312 on 2 and 27 DF,  p-value: 0.05172
```

**d)** The Bonferroni method should be chosen because the Šidák method is based on the assumption of independence of hypotheses. In the considered case, the hypotheses are mostly likely dependent. $H_0$ is rejected if the corresponding $p$-value is less than $\alpha_{\text{loc}}$ which is equal to the family wise error rate (FWER) devided by the number of hypotheses. In our case

$$\alpha_{\text{loc}} = \frac{0.05}{3} = 0.017.$$

Thus the first two null hypotheses are rejected while the third one is not rejected.

**3.**

**a)** Since $EX = 0$, using the trace formula, we obtain

$$EQ = \text{tr}(\Sigma^{-1}\Sigma) = \text{tr}(I_p) = p.$$

**4.**

**a)** These matrices are idempotent. This is known from our course or can be easily obtained directly. But the only invertible idempotent matrix is the identity matrix. Indeed, let $A$ be an invertible idempotent matrix. Multiplying both sides of the equality

$$AA = A$$

by $A^{-1}$, we obtain

$$A = I.$$

If $H$ is invertible and therefore $H = I$, then the two matrices coincide.

**5.**

**a)** Multiplying both sides of the equality $X_1+X_2+X_3 = 0$ by $X_1$ and taking the expectation, we obtain

$$E[X_1(X_1 + X_2 + X_3)] = 0$$

i.e.

$$1 + \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) = 0.$$

Similarly (multiplying by $X_2$ and $X_3$) we obtain two more equations:

$$\mathrm{Cov}(X_1, X_2) + 1 + \mathrm{Cov}(X_2, X_3) = 0,$$

$$\mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_2, X_3) + 1 = 0.$$

Solving these three equations, we obtain

$$\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = -\frac{1}{2}.$$

Hence, the covariance matrix of $X$ is

$$\Sigma = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

**b)** $Y$ is normal (linear transformation of the normal vector $X$), $EY = 0$, the covariance matrix is

$$\mathrm{Cov}(Y) = A\Sigma A^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix},$$

therefore the characteristic function is

$$\phi(t_1, t_2) = e^{-\frac{1}{2}(t_1^2 - t_1 t_2 + t_2^2)}.$$