Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4267 Linear Statistical Models**

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Examination date: 24 May 2023

Examination time (from–to): 09:00 – 13:00

Permitted examination support material: C:
– Tabeller og formler i statistikk, Tapir forlag,
– K.Rottman. Matematisk formelsamling,
– Stamped yellow A4 sheet with your own handwritten notes,
– Calculator: HP30S, Citizen SR-270X, Citizen SR-270X College or Casio fx-82ES PLUS.

Other information:
Sensur:

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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skal ha flervalgskjema ☐	

Date Signature

Problem 1 Assume that $X = (X_1, X_2)^T$ is a bivariate normal random vector with

$$\mu = EX = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \Sigma = \mathrm{Cov}(X) = \begin{pmatrix} 13/5 & 1 \\ 1 & 13/5 \end{pmatrix}.$$

a) Find a symmetric 2×2 matrix A with equal diagonal elements such that components of the random vector $Y = AX$ are independent and

$$EY = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

b) Find a 2×2 matrix C such that random vectors CX and X are independent.

Problem 2 A multiple linear regression model is considered. It is assumed that

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i, \ i = 1, ..., n,$$

where $\epsilon = (\epsilon_1, ..., \epsilon_n)^T$ has the normal distribution with zero expectation and covariance matrix $\sigma^2 I$. Suppose that $n = 30$, $x_{i1} = 1$ for $1 \leq i \leq 20$ and 0 otherwise, $x_{i2} = 0$ for $1 \leq i \leq 10$ and 1 otherwise. Denote the least squares estimator of $\beta = (\beta_0, \beta_1, \beta_2)^T$ by $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)^T$.

a) Find the correlation coefficient between $\hat{\beta}_1$ and $\hat{\beta}_2$. The following information (R input and output) might be useful:

```
> matrix<-array(c(30,20,20,20,20,10,20,10,20),dim = c(3,3))
> inverse<-solve(matrix)
> inverse
     [,1] [,2] [,3]
[1,]  0.3 -0.2 -0.2
[2,] -0.2  0.2  0.1
[3,] -0.2  0.1  0.2
```

b) Perform a test for

$$H_0 : \beta_1 = 2\beta_2$$

vs.

$$H_1 : \beta_1 \neq 2\beta_2.$$

The significance level is 0.05. Use the following information (R output):

```
Call:
lm(formula = Y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-4.1806 -1.2880  0.3316  1.3483  4.7308

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)        ?      1.228   0.776    0.445
x1             2.579          ?   2.571    0.016 *
x2             1.196      1.003       ?    0.243
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.243 on 27 degrees of freedom
Multiple R-squared:  0.197,     Adjusted R-squared:  0.1375
F-statistic: 3.312 on 2 and 27 DF,  p-value: 0.05172
```

c) In the R output three numerical values are replaced by question marks. Explain the three columns containing question marks and calculate numerical values.

d) Now suppose that the following three hypotheses are tested simultaneously:

$$H_0 : \beta_1 = \beta_2 - 1 \text{ vs. } H_1 : \beta_1 \neq \beta_2 - 1,$$

$$H_0 : \beta_1 = \beta_2 \text{ vs. } H_1 : \beta_1 \neq \beta_2,$$

and

$$H_0 : \beta_1 = \beta_2 + 1 \text{ vs. } H_1 : \beta_1 \neq \beta_2 + 1.$$

Probability of at least one Type I error must not be greater than 0.05. One of the following two methods can be used: the Bonferroni method and the Šidák method. Which one do you choose? Why? Which null hypotheses are rejected if the p-values are given in the table below? Why?

H_0	$\beta_1 = \beta_2 - 1$	$\beta_1 = \beta_2$	$\beta_1 = \beta_2 + 1$
p-value	0.002	0.014	0.571

Problem 3 Let X be a p-variate random vector with zero mean and nonsingular covariance matrix Σ.

a) Find the expectation of the quadratic form

$$Q = X^T \Sigma^{-1} X.$$

Problem 4 The classical multiple linear regression model (including an intercept) is written in matrix form as

$$Y = X\beta + \epsilon.$$

The hat matrix is $H = X(X^T X)^{-1} X^T$.

a) Prove that the centering matrix

$$C = I - \frac{1}{n} 1 1^T,$$

and the matrix

$$H - \frac{1}{n} 1 1^T$$

are non-invertibe. Here 1 is the vector of ones. What is the relationship between these two matrices if the hat matrix is invertible?

Problem 5 Let X_1, X_2, X_3 be univariate random variables with zero expectations and unit variances. Their sum is equal to zero: $X_1 + X_2 + X_3 = 0$.

a) Find the covariance matrix Σ of the random vector $X = (X_1, X_2, X_3)^T$.

Suppose now that X is normal. Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $Y = (Y_1, Y_2)^T = AX$.

b) Find the characteristic function of Y.