Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4267 Linear statistical models
Recommended exercises 1

Problem 1 Simple matrix calculations

Solve the problems by hand *and* by use of R (when possible).

Let $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$.

a) Construct A as a matrix in R. Use the function `matrix`.

b) Is A symmetric? Use the R function `t`.

c) Show that A is positive definite (not in R – use the definition of positive definiteness).

d) Find the eigenvalues and the eigenvectors of A. Are the eigenvectors found by R normalized? Use the function `eigen`.

e) Find an orthogonal diagonalization of A. In R, matrix multiplication is performed by `%*%`.

f) Find A^{-1}. Use the R function `solve`.

g) Find the eigenvalues and the eigenvectors of A^{-1}. Is there a relationship between the eigenvalues and the eigenvectors of A and A^{-1}?

h) Why can A be a covariance matrix?

i) Assume that A is the covariance matrix of a random vector. Find the correlation matrix, that is, the matrix having the correlation coefficient of the i and j entries of the random vector as its ij entry The R functions `diag` and `sqrt` may be useful. Check your computations with `cov2cor`.

j) Let $\boldsymbol{X}$ be a random vector, and assume

$$E\boldsymbol{X} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \operatorname{Cov} \boldsymbol{X} = A.$$

Find, in R, the expectation and covariance matrices of

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{X}, \qquad \begin{pmatrix} 1 & 2 \end{pmatrix} \boldsymbol{X} \qquad \text{and} \qquad \begin{pmatrix} \boldsymbol{X} \\ 3\boldsymbol{X} \end{pmatrix}$$

(the last is a block matrix, in this case the concatenation of the vectors $\boldsymbol{X}$ and $3\boldsymbol{X}$).

Problem 2 Mean and covariance of linear combinations

Let $\boldsymbol{X}$ be a trivariate (three-dimensional) random vector with mean (expectation) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}^{\mathrm{T}}$ and covariance matrix I, a 3×3 identity matrix. Find the mean and covariance matrix of

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \boldsymbol{X}.$$

Problem 3 Covariance formula

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be random vectors of the same dimension. The covariance matrix of $\boldsymbol{V}$ and $\boldsymbol{W}$ is defined as $\operatorname{Cov}(\boldsymbol{V}, \boldsymbol{W}) = E\Big((\boldsymbol{V} - E\boldsymbol{V})(\boldsymbol{W} - E\boldsymbol{W})^{\mathrm{T}}\Big)$. Show that $\operatorname{Cov}(\boldsymbol{V}, \boldsymbol{W}) = E(\boldsymbol{V}\boldsymbol{W}^{\mathrm{T}}) - (E\boldsymbol{V})(E\boldsymbol{W})^{\mathrm{T}}$.

Note that this is a generalization of a well-known formula for univariate variables. In the case $\boldsymbol{W} = \boldsymbol{V}$, we get $\operatorname{Cov}(\boldsymbol{V}) = E(\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}) - (E\boldsymbol{V})(E\boldsymbol{V})^{\mathrm{T}}$, which is a generalization of the univariate $\operatorname{Var} V = EV^2 - (EV)^2$.

Problem 4 The square root matrix and the Mahalanobis transform

Let the expectation (mean) and covariance matrix of a random vector $\boldsymbol{X}$ be $\boldsymbol{\mu} = E\boldsymbol{X}$ and $\Sigma = \operatorname{Cov} \boldsymbol{X}$. Let P be an orthogonal matrix having eigenvectors of Σ as columns and Λ a diagonal matrix having the eigenvalue corresponding to the ith column of Σ as its ii entry. Then $\Sigma = P\Lambda P^{\mathrm{T}}$.

a) Show that Σ is positive semidefinite. (A symmetric matrix A is positive semidefinite if $\boldsymbol{z}^{\mathrm{T}} A \boldsymbol{z} \geq 0$ for all vectors $\boldsymbol{z}$.)

Assume that Σ is positive definite. (A symmetric matrix A is positive definite if $\boldsymbol{z}^{\mathrm{T}} A \boldsymbol{z} > 0$ for all vectors $\boldsymbol{z} \neq \boldsymbol{0}$.)

b) Show that all eigenvalues of Σ are positive.

Why does Σ have an inverse? What can you say about the eigenvalues and eigenvectors of Σ^{-1}? Justify the answer.

c) Let $\Lambda^{1/2}$ be the diagonal matrix having as entries the square root of those of Λ, and let $\Lambda^{-1/2} = (\Lambda^{1/2})^{-1}$. Define

$$\Sigma^{1/2} = P\Lambda^{1/2}P^{\mathrm{T}} \qquad \text{and} \qquad \Sigma^{-1/2} = P\Lambda^{-1/2}P^{\mathrm{T}}.$$

Show that both are symmetric, and that

$$\Sigma^{1/2}\Sigma^{1/2} = \Sigma, \qquad \Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1} \qquad \text{and} \qquad \Sigma^{1/2}\Sigma^{-1/2} = I,$$

where I is an identity matrix.

d) The transform $\boldsymbol{Y} = \Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})$ is called the Mahalanobis transform. Show that $E\boldsymbol{Y} = \boldsymbol{0}$ and $\operatorname{Cov} \boldsymbol{Y} = I$.