Norwegian University of Science and Technology
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TMA4267 Linear statistical models
Recommended exercises 4 – solutions

Problem 1 Symmetric, idempotent matrices

a) $\left(\begin{smallmatrix} 1 & 0 \\ c & 0 \end{smallmatrix}\right)$ is idempotent but not symmetric for $c \neq 0$.

b) $\operatorname{rank} A = \operatorname{rank} \Lambda = \operatorname{tr} \Lambda = \operatorname{tr} A$. The first and third equalities follow from information given in the problem. The second follow from the fact that $\operatorname{tr} \Lambda$ is the number of non-zero columns of Λ (remember that the eigenvalues of A are 0 and 1), which is also $\operatorname{rank} A$.

c)
$$\left(\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)^{\mathrm{T}} = \frac{1}{n}(\mathbf{1}^{\mathrm{T}})^{\mathrm{T}}\mathbf{1}^{\mathrm{T}} = \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}},$$
showing that $\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is symmetric.
$$\left(\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)^{2} = \frac{1}{n}\frac{1}{n}\mathbf{1}\underbrace{\mathbf{1}^{\mathrm{T}}\mathbf{1}}_{=n}\mathbf{1}^{\mathrm{T}} = \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}},$$
showing that $\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is idempotent. $\operatorname{rank} \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} = \operatorname{tr} \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} = n \cdot \frac{1}{n} = 1$.

In general, if A is symmetric, then $I - A$ is symmetric, since, in that case, $(I - A)^{\mathrm{T}} = I^{\mathrm{T}} - A^{\mathrm{T}} = I - A$. If A is idempotent, then so is $I - A$, since, in that case, $(I - A)^2 = (I - A)(I - A) = I^2 - AI - IA + A^2 = I - A - A + A = I - A$. So $I - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is symmetric and idempotent. Finally, $\operatorname{rank}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}) = \operatorname{tr}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}) = \operatorname{tr} I - \operatorname{tr}(\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}) = n - 1$.

Problem 2 Linear combinations and quadratic forms

a) By the trace formula,
$$E(\boldsymbol{X}^{\mathrm{T}}A\boldsymbol{X}) = \operatorname{tr}(AI) + \begin{pmatrix}1 & 1 & 1\end{pmatrix} A \begin{pmatrix}1 & 1 & 1\end{pmatrix}^{\mathrm{T}} = \operatorname{tr} A + \begin{pmatrix}1 & 1 & 1\end{pmatrix} \mathbf{0} = 3 \cdot \frac{2}{3} = 2.$$

b) It is easy to verify that $A^{\mathrm{T}} = A$, so that A is symmetric, and that $A^2 = A$, so that A is idempotent. The rank of an idempotent matrix is equal to the trace, the sum of the diagonal entries, so $\operatorname{rank} A = \operatorname{tr} A = 2$.

We know that for $Y \sim N(\mathbf{0}, \sigma^2)$, the quadratic form $\boldsymbol{Y}^{\mathrm{T}}A\boldsymbol{Y} \sim \chi^2_r$, where r is the rank of the symmetric idempotent matrix A. So $(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}A(\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi^2_2$, where $\boldsymbol{\mu} = (1\ 1\ 1)^{\mathrm{T}}$. But $A(\boldsymbol{X} - \boldsymbol{\mu}) = A\boldsymbol{X} - A\boldsymbol{\mu} = A\boldsymbol{X}$ and $(\boldsymbol{X} - \mu)^{\mathrm{T}}A(\boldsymbol{X} - \boldsymbol{\mu}) = (\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}A\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}A\boldsymbol{X} - (A\mu)^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}A\boldsymbol{X}$, so $\boldsymbol{X}^{\mathrm{T}}A\boldsymbol{X} = (\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}A(\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi^2_2$.

According to statistical tables, the 0.95-quantile of a χ^2_2-variable is 5.991, thus the probability that the quadratic form is less than 6 is approximately 0.95.

Problem 3 The F-distribution

a) The joint distribution of V and W is the product of the two marginal distributions.

$$f_{V,W}(v,w) = \frac{1}{2^{p/2}\Gamma(p/2)} v^{p/2-1} e^{-v/2} \cdot \frac{1}{2^{q/2}\Gamma(q/2)} w^{q/2-1} e^{-w/2}.$$

The inverse of the transformation $f = (v/p)/(w/q)$, $g = w$ is given by $v = pfg/q$, $w = g$, with Jacobian

$$\begin{vmatrix} pg/q & pf/q \\ 0 & 1 \end{vmatrix} = \frac{p}{q} g.$$

Then the joint distribution of F and G is given by

$$\begin{aligned} f_{F,G}(f,g) &= f_{V,W}\left(\frac{p}{q} fg, g\right) \cdot \left|\frac{p}{q} g\right| \\ &= \frac{1}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)} \left(\frac{p}{q} fg\right)^{p/2-1} g^{q/2-1} e^{-(1+pf/q)g/2} \cdot \frac{p}{q} g \\ &= \frac{(p/q)^{p/2} f^{p/2-1}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)} g^{(p+q)/2-1} e^{-(1+pf/q)g/2}. \end{aligned}$$

Then the marginal distribution of F is given by

$$\begin{aligned} f_F(f) &= \frac{(p/q)^{p/2} f^{p/2-1}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)} \int_0^\infty g^{(p+q)/2-1} e^{-(1+pf/q)g/2}\, dg \\ &\qquad\qquad (u = (1+pf/q)g,\ \ du = (1+pf/q)\, dg) \\ &= \frac{(p/q)^{p/2} f^{p/2-1}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)} \int_0^\infty \frac{u^{(p+q)/2-1}}{(1+pf/q)^{(p+q)/2-1}} e^{-u/2} \frac{du}{1+pf/q} \\ &= \frac{(p/q)^{p/2} f^{p/2-1}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)(1+pf/q)^{(p+q)/2}} \int_0^\infty u^{(p+q)/2-1} e^{-u/2} du \\ &= \frac{\Gamma((p+q)/2)(p/q)^{p/2} f^{p/2-1}}{\Gamma(p/2)\Gamma(q/2)(1+pf/q)^{(p+q)/2}} \int_0^\infty \frac{1}{2^{(p+q)/2}\Gamma((p+q)/2)} u^{(p+q)/2-1} e^{-u/2} du \\ &\qquad\qquad (\text{integrand is pdf of } \chi^2_{p+q} \text{ variable}) \\ &= \frac{\Gamma((p+q)/2)(p/q)^{p/2}}{\Gamma(p/2)\Gamma(q/2)} \frac{f^{p/2-1}}{(1+pf/q)^{(p+q)/2}}. \end{aligned}$$

b) If $V \sim \chi^2_p$ and $W \sim \chi^2_q$ and V and W are independent, $F = \frac{V/p}{W/q} \sim F_{p,q}$. Then $1/F = \frac{W/q}{V/p} \sim F_{q,p}$ by definition.

c) If $Z \sim N(0,1)$ and $V \sim \chi^2_q$ and Z and V are independent, $T = Z/\sqrt{V/q} \sim t_q$. Then $T^2 = \frac{Z^2/1}{V/q} \sim F_{1,q}$ by definition (remember that $Z^2 \sim \chi^2_1$).

Problem 4 **T- and F-distributions in R**

```
## a

B <- 10000
p <- 9
u <- rnorm(B) # draw B standard normal variates
v <- rchisq(B, p) # draw B chi^2 variates with df = p
t <- u / sqrt(v / p)
hist(t, nclass = 50, freq = FALSE)
# freq = FALSE: probabilities at y-axis
plot(function(x)
  dt(x, df = p),
  min(t),
  max(t),
  add = TRUE,
  col = "red")
abline(v = qt(0.05, p), col = "green")
abline(v = qt(0.95, p), col = "green")

## b

f <- t ^ 2
hist(f, nclass = 50, freq = FALSE)
plot(function(x)
  df(x, 1, p),
  min(f),
  max(f),
  add = TRUE,
  col = "red")
# df: pdf of F-distribution
abline(v = qf(0.05, 1, p), col = "green")
abline(v = qf(0.95, 1, p), col = "green")

## c

n1 <- 5
n2 <- 40
u <- rchisq(B, df = n1)
v <- rchisq(B, df = n2)
```

```
f <- u / n1 / (v / n2)
hist(f, nclass = 50, freq = FALSE)
plot(function(x)
  df(x, n1, n2),
  min(f),
  max(f),
  add = TRUE,
  col = "red")
abline(v = qf(0.05, n1, n2), col = "green")
abline(v = qf(0.95, n1, n2), col = "green")
```