Solutions (TMA4267 2024 May)

1.

$$F_Y(x, y) = P(X \le x, X \le y) = P(X \le \min\{x, y\}) = F(\min\{x, y\}).$$

For Z. If $y \le 0$ then $P(e^X \le y)) = 0$ and therefore $F_Z(x, y) = 0$. Suppose that $y > 0$. Then

$$F_Z(x, y) = P(X \le x, E^X \le y) = P(X \le x, X \le \ln y) =$$

$$= P(X \le \min\{x, \ln y\}) = F(\min\{x, \ln y\}).$$

Thus

$$F_Z(x, y) = \begin{cases} 0 \text{ if } y \le 0, \\ F(\min\{x, \ln y\}) \text{ if } y > 0. \end{cases}$$

2. Since AZ and BZ are independent, their covariance matrix is zero:

$$\mathrm{Cov}(AZ, BZ) = AIB^T = AB = \begin{pmatrix} 1 + ab & b + a \\ a + b & ab + 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e. $ab + 1 = 0$ and $a + b = 0$. There are two solutions $a = 1, b = -1$ and $a = -1, b = 1$.

3.

a) Since $p = 3$ and $n - p$ is 17 (the numbers of degrees of freedom of the F-statistic are $p - 1$ and $n - p$), the sample size is $n = 20$.

`Estimate`

is $\hat{\beta}_j$ $(j = 0, 1, 2)$;

`Std. Error`

is $\sqrt{\widehat{\mathrm{Var}\hat{\beta}_j}}$;

`t value`

is $t_j = \hat{\beta}_j / \sqrt{\widehat{\mathrm{Var}\hat{\beta}_j}}$.

`Adjusted R-squared`

is

$$R^2_{\text{adj}} = 1 - (1 - R^2)\frac{n - 1}{n - k - 1}.$$

Thus, for the first question mark:

$$\sqrt{\widehat{\mathrm{Var}\hat{\beta}_0}} = \hat{\beta}_0 / t_0 = 0.2119/0.518 = 0.4091,$$

for the second:

$$t_1 = \hat{\beta}_1 / \sqrt{\widehat{\text{Var}\hat{\beta}_1}} = 1.2285/0.317 = 3.875,$$

for the third:

$$\hat{\beta}_2 = t_2 \cdot \sqrt{\widehat{\text{Var}\hat{\beta}_2}} = 6.952 \cdot 0.317 = 2.2037,$$

for the fourth:

$$R_{\text{adj}} = 1 - (1 - 0.7502)\frac{19}{17} = 0.7208.$$

Thus the output is in fact as follows

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Call:
lm(formula = Y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-0.8819 -0.3581  0.1598  0.3356  1.0157

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.2119     0.4091  -0.518  0.61123
x1            1.2285     0.3170   3.875  0.00121 **
x2            2.2036     0.3170   6.952 2.34e-06 ***
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Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.5787 on 17 degrees of freedom
Multiple R-squared:  0.7502,    Adjusted R-squared:  0.7208
F-statistic: 25.53 on 2 and 17 DF,  p-value: 7.576e-06
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The second and third diagonal elements of the matrix $(X^TX)^{-1}$ are equal because the second and third elements of column

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Std. Error
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are equal.

b) The covariance matrix of $\hat{\beta}$ is $\sigma^2(X^TX)^{-1}$. In our case

$$X^TX = \begin{bmatrix} 20 & 15 & 15 \\ 15 & 15 & 10 \\ 15 & 10 & 15 \end{bmatrix},$$

$$(X^TX)^{-1} = \begin{bmatrix} 0.5 & -0.3 & -0.3 \\ -0.3 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0.3 \end{bmatrix},$$

therefore the correlation coefficient between $\hat{\beta}_1$ and $\hat{\beta}_2$ is

$$\frac{\text{Cov}(\hat{\beta}_1, \hat{\beta}_2)}{\sqrt{\text{Var}(\hat{\beta}_1)}\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\sigma^2 \cdot 0.1}{\sqrt{\sigma^2 \cdot 0.3}\sqrt{\sigma^2 \cdot 0.3}} = \frac{1}{3}.$$

c) $(1-\alpha)$-confidence interval for β_j $(j = 0, 1, 2)$ is

$$\left[\hat{\beta}_j - t_{\frac{\alpha}{2}, n-p}\sqrt{\widehat{\text{Var}\hat{\beta}_j}}, \hat{\beta}_j + t_{\frac{\alpha}{2}, n-p}\sqrt{\widehat{\text{Var}\hat{\beta}_j}}\right].$$

In our case

$$\hat{\beta}_0 = -0.2119, \ \sqrt{\widehat{\mathrm{Var}\hat{\beta}_0}} = 0.4092, \ t_{0.025,17} = 2.11,$$

therefore the confidence interval is $[-1.0753, 0.6515]$.

d) Consider the followin matrix: $A = [0\ 1\ -1]$. The null hypothesis and alternative can be written as

$$H_0 : A\beta = 0 \text{ vs. } H_1 : A\beta \neq 0.$$

We use the general F-test. Test statistic is

$$F = \frac{(A\hat{\beta})^T (A(X^T X)^{-1} A^T)^{-1} A\hat{\beta}}{\hat{\sigma}^2}.$$

Under H_0 this statistic has F-distribution with 1 and $n-p$ degrees of freedom. In our case $n = 20$, $p = 3$, $\hat{\beta}_1 = 1.2285$, $\hat{\beta}_2 = 2.2036$, $\hat{\sigma} = 0.5787$,

$$A(X^T X)^{-1} A^T = 0.4,$$

$$(A(X^T X)^{-1} A^T)^{-1} = \frac{10}{4} = 2.5,$$

$$A\hat{\beta} = \hat{\beta}_1 - \hat{\beta}_2 = -0.9751,$$

$$(A\hat{\beta})^T (A(X^T X)^{-1} A^T)^{-1} A\hat{\beta} = 0.9751^2 \cdot 2.5 = 2.38,$$

$$F = \frac{2.38}{0.5787^2} = 7.1.$$

Thus the observed value of the test statistic is $F = 7.1$. Since $f_{0.05,1,17} = 4.45$, the null hypothesis is rejected.

e) According to the Šidák method, H_0 is rejected if the corresponding p-value is smaller than α_{loc}, where

$$\alpha_{\mathrm{loc}} = 1 - (1 - \mathrm{FWER})^m,$$

and m is the number of hypotheses. In our case $m = 3$, FWER $= 0.05$, $\alpha_{\mathrm{loc}} = 0.017$. The p-values are in the column

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Pr(>|t|)
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They are 0.61123, 0.00121, 2.34e-06. Therefore the first hypothesis is not rejected, the second and the third are rejected.

f) It is known from the course that if Z is a n-variate standard normal vector and R is a $n \times n$ symmetric idempotent matrix such that $\mathrm{rank}(R) = r$, then the quadratic form $Z^T R Z$ has chi-square distribution with r degrees of freedom. In our case, $Y - X\beta$ has the n-variate standard normal distribution, the matrix $H = X(X^T X)^{-1} X^T$ (hat matrix) is symmetric idempotent, $\mathrm{rank}(H) = p$. Therefore the distribution of the quadratic form is chi-square with p degrees of freedom. Thus the expectation and the variance of the quadratic form are p and $2p$. Since $p = 3$, we finally obtain 3 and 6.

4. It is known from the course that

$$g(y) = c \int \frac{dy}{\sqrt{h(y)}},$$

where c is an arbitrary constant, and $h(y)$ is such a function that

$$\mathrm{Var}Y = h(EY).$$

In our case (chi square distribution), $EY = m$, $\mathrm{Var}Y = 2m$, therefore $h(y) = 2y$, and

$$g(y) = \frac{c}{\sqrt{2}} \int \frac{dy}{\sqrt{y}} = \sqrt{2}c\sqrt{y}.$$

Take for simplicity $c = 1/\sqrt{2}$. Then $g(y) = \sqrt{y}$.

5. Note that for non-negative numbers a and b, the following equality holds:

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}.$$

Therefore

$$E \max\{X_1^2, X_2^2\} = E\frac{X_1^2 + X_2^2 + |X_1^2 - X_2^2|}{2} = \frac{EX_1^2 + EX_2^2}{2} +$$

$$+\frac{1}{2}E(|X_1 - X_2| \cdot |X_1 + X_2|) = 1 + \frac{1}{2}E(|X_1 - X_2| \cdot |X_1 + X_2|) \leq$$

(the Cauchy-Schwarz inequality)

$$\leq 1 + \frac{1}{2}\sqrt{E(X_1 - X_2)^2 E(X_1 + X_2)^2} =$$

$$= 1 + \frac{1}{2}\sqrt{(EX_1^2 + EX_2^2 - 2EX_1X_2)(EX_1^2 + EX_2^2 + 2EX_1X_2)} =$$

$$= 1 + \sqrt{(1 - \rho)(1 + \rho)} = 1 + \sqrt{1 - \rho^2}.$$