Solution FMA 4267 V-2011

1

a) $\underline{Y} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underline{A}\,\underline{X}$ and therefore binormal-distributed since $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is binormal distributed.

$$\mu_{\underline{Y}} = \begin{bmatrix} \mu(1+k) \\ \mu \end{bmatrix}. \quad \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{Cov(X_1, X_2)}{\sigma k \sigma} = \rho \Rightarrow Cov(X_1, X_2) = \rho k \sigma^2.$$

$$\Sigma_{\underline{X}} = \sigma^2 \begin{bmatrix} 1 & \rho k \\ \rho k & k^2 \end{bmatrix} \Rightarrow \Sigma_{\underline{Y}} = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \rho k \\ \rho k & k^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 + 2\rho k + k^2 & 1 + \rho k \\ 1 + \rho k & 1 \end{bmatrix}$$

b) $E[X_1 + X_2 \mid X_1 = x_1] = \mu(1+k) + \frac{\sigma^2(\rho k + 1)}{\sigma^2}(x_1 - \mu)$

$= \mu(1+k) + (\rho k + 1)(x_1 - \mu)$

$k = 4, \ \rho = \frac{1}{2}, \ x_1 - \mu = 1 \Rightarrow E[X_1 + X_2 \mid X_1 = \mu + 1] = 5\mu + 3$

$k = 1$ og $\rho = \frac{1}{2} \Rightarrow \Sigma_{\underline{X}} = \sigma^2 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda' \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \Rightarrow \begin{aligned} e_1 + \tfrac{1}{2} e_2 &= \lambda' e_1 \quad (1) \\ \tfrac{1}{2} e_1 + e_2 &= \lambda' e_2 \quad (2) \end{aligned}, \quad \lambda' = \frac{\lambda}{\sigma^2}$$

(1) $\Rightarrow e_1 = \frac{e_2}{2(\lambda' - 1)}$ which inserted in (2) gives

$\frac{e_2}{4(\lambda'-1)} + e_2 = \lambda' e_2 \Rightarrow \lambda'^2 - 2\lambda' + \frac{3}{4} = 0 \Rightarrow \lambda = \begin{cases} \frac{3}{2} \\ \frac{1}{2} \end{cases} \Rightarrow \lambda = \begin{cases} \frac{3\sigma^2}{2} \\ \frac{\sigma^2}{2} \end{cases}$

$\lambda' = \frac{3}{2} \Rightarrow e_2 = e_1$ and the 1. principal component is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

which explain $\frac{\frac{3}{2}}{2} = 75\%$ of $Var(X_1 + X_2)$

2a). $F_{obs} = \dfrac{\frac{SS_R}{3}}{\frac{SS_E}{14}} = 69.12.$

$P(F_{3,14} \geq 69.12) = 1.236 \cdot 10^{-8}$ shows that the regression is significant.

Multiple R-squared $= \dfrac{SS_R}{SS_T}$ and tells how much of the variation in the response that is explained by the model.

$s = \sqrt{\frac{SS_E}{14}} = 19.63 \Rightarrow SS_E = (19.63)^2 \cdot 14 = 5394.72.$

From F_{obs} we have $SS_R = \dfrac{SS_E}{14} \cdot 69.12 \cdot 3 = (19.63)^2 \cdot 69.12 \cdot 3$

$= 79903.46$

b) $SS_E^2 = (14.59)^2 \cdot 12 = 2554.42.$

$E(Y) = \beta_0 + \beta_1 ph + \beta_2 z_1 + \beta_3 z_3 + \beta_4 ph z_1 + \beta_5 ph z_2$

$H_0: \beta_3 = \beta_5 = 0$ $\quad H_1:$ minst ein $\neq 0$.

$$F_{2,12} = \frac{\frac{SS_R(FM) - SS_R(RM)}{2}}{\frac{SS_E(FM)}{12}} = \frac{\frac{SS_E(RM) - SS_E(FM)}{2}}{\frac{SS_E(FM)}{12}}$$

$$= \frac{\frac{5394.72 - 2554.42}{2}}{\frac{2554.42}{12}} = 6.67 > 3.89 = q_{005}(2, 12)$$ i.e.

at least one of the terms with z_2 should be in the model

c) $E(y) = \beta_0 + \beta_1 ph + \beta_2 z_1 + \beta_3 ph\,z_1 + \beta_4 ph\,z_2$

$H_0: \beta_4 = 0 \qquad H_1: \beta_4 \neq 0.$

$t_{obs} = 3.16. \qquad P(|T_{13}| \geq 3.16) = 0.00755$ which implies that $ph\,z_2$ is significal at a 5% level given the other variables.

Polymer 3 $\qquad z_1 = z_2 = 0$

$$\hat{y} = -208.98 + 60.31\,ph$$

Polymer 2 $\qquad z_1 = 0, \quad z_2 = 1$

$$\hat{y} = -208.98 + 60.31\,ph + 3.58\,ph$$

$$= -208.98 + 63.89\,ph.$$

Polymer 1

$$\hat{y} = -208.98 + 60.31\,ph + 248.395 - 20.05\,ph$$

$$= 39.415 + 40.16\,ph.$$

3a) $\hat{\beta} = (\underline{X}^t\underline{X})^{-1}\underline{X}^t(\underline{X}\underline{\beta} + \underline{\varepsilon}) = \underline{\beta} + (\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{\varepsilon}$

such that $\hat{\underline{\beta}} - \underline{\beta} = (\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{\varepsilon}$

$Cov(\hat{\beta}) = E[(\hat{\underline{\beta}} - \underline{\beta})(\hat{\underline{\beta}} - \underline{\beta})^t] = E[(\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{\varepsilon}\,\underline{\varepsilon}^t\underline{X}(\underline{X}^t\underline{X})^{-1}] = \sigma^2(\underline{X}^t\underline{X})^{-1}$

$\underline{e} = \underline{Y} - \underline{X}\hat{\underline{\beta}} = \underline{Y} - \underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{Y} = (\underline{I} - \underline{H})\underline{Y}$

$= (\underline{I} - \underline{H})(\underline{Y} - \underline{X}\underline{\beta}) = (\underline{I} - \underline{H})\underline{\varepsilon}$ since $(\underline{I} - \underline{H})(\underline{Y} - \underline{X}\underline{\beta}) = (\underline{I} - \underline{H})\underline{Y} - \underline{X}\underline{\beta} + \underline{X}\underline{\beta}$

$= (\underline{I} - \underline{H})\underline{Y}$

b) $SS_E = \underline{e}^t\underline{e} = \underline{\varepsilon}^t(\underline{I} - \underline{H})^t(\underline{I} - \underline{H})\underline{\varepsilon} = \underline{\varepsilon}^t(\underline{I} - \underline{H})\underline{\varepsilon}$ since $(\underline{I} - \underline{H})$ is symmetric and idempotent.

$E[SS_E] = tr[(\underline{I} - \underline{H}) \cdot \sigma^2\underline{I}] + \underline{0}^t(\underline{I} - \underline{H})\underline{0} = \sigma^2 tr(\underline{I} - \underline{H})$

$= \sigma^2(tr(\underline{I}) - tr(\underline{H}))$ (H idempotent) $= \sigma^2(n - rank(H)) = \sigma^2(n - (k+1))$

$\hat{\sigma}^2 = SS_E/(n-k-1)$

c) $\frac{SS_E}{\sigma^2} = \frac{\underline{\varepsilon}^t(I-H)\underline{\varepsilon}}{\sigma^2} \sim \chi^2(rank(I-H))$ since $\underline{\varepsilon} \sim N(\underline{0}, \sigma^2\underline{I})$

$rank(\underline{I} - \underline{H}) = tr(\underline{I} - \underline{H})$ since $\underline{I} - \underline{H}$ is idempotent.

$tr(\underline{I} - \underline{H}) = n - k - 1.$

$Cov(\hat{\underline{\beta}}, \underline{e}) = E[(\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{\varepsilon}\,\underline{\varepsilon}^t(\underline{I} - \underline{H})] = (\underline{X}^t\underline{X})^{-1}\underline{X}^t\sigma^2\underline{I}(\underline{I} - \underline{H})$

$= \sigma^2(\underline{X}^t\underline{X})^{-1}\underline{X}^t(\underline{I} - \underline{H}) = \sigma^2[(\underline{X}^t\underline{X})^{-1}\underline{X}^t - (\underline{X}^t\underline{X})^{-1}\underline{X}^t\underline{X}(\underline{X}^t\underline{X})^{-1}\underline{X}^t] = \sigma^2\underline{0}$

Since $\hat{\underline{\beta}}$ and $\underline{e}$ are independent it follows that $\hat{\underline{\beta}}$ and $\underline{e}^t\underline{e}$ also are independent.