Solution TMA 4267, May 25 2013

Problem 1

a) $\rho_{X_3, X_1} = \frac{1}{\sqrt{3}\cdot\sqrt{1}} = \frac{\sqrt{3}}{3}$

$\rho_{X_3 X_2} = \frac{-1}{\sqrt{2}\sqrt{3}} = -\frac{\sqrt{6}}{6}$, since $\left|\frac{\sqrt{3}}{3}\right| > \left|-\frac{\sqrt{6}}{6}\right|$

if follow that X_1 have the strongest correlation with X_3 in absolute value

Let $\underline{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then $\underline{Z} = \underline{A}\underline{X}$, i.e. a linear transformation of $\underline{X}$ which is trivariate normal and thereby bivariate normal distributed

since $\underline{A}$ is a 2×3 matrix

$$E[\underline{Z}] = \underline{A}\underline{\mu} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_1 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Sigma_Z = \underline{A}\underline{\Sigma}\underline{A}^t = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

b) $\underline{Y} = \begin{bmatrix} \underline{e}_1^t \\ \underline{e}_2^t \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underline{A}\underline{X}$ and is thereby a linear transformation

of $\underline{X} \sim N_3(\underline{\mu}, \underline{\Sigma})$. $\underline{A}$ is a 2×3 matrix

$$\mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(\underline{e}_1^t\underline{X}, \underline{e}_2^t\underline{X}) = E[\underline{e}_1^t(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^t\underline{e}_2]$$

$$= \underline{e}_1^t\underline{\Sigma}\underline{e}_2 = \underline{e}_1^t\lambda\underline{e}_2 = \lambda\underline{e}_1^t\underline{e}_2 = 0$$

Since Y_1 and Y_2 are univariate normal and uncorrelated, they are independent.

$$\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} = \frac{3.88+1.65}{6} = 0.921, \text{ i.e about } 92.1\%$$

of the total variation in $\underline{X}$ can be explained by $\underline{Y}$.

c) Since $\underline{Y}$ is bivariate normal distributed we know that D is χ^2-distributed with two degrees of freedom. If $D_{obs} \geq \chi^2_{\alpha,2}$ for some α the product would not pass the control.

Suppose $\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is trivariate normal distributed with mean vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \underline{\Lambda}$

Let $\underline{X} = \underline{P}\,\underline{U} = [\underline{e}_1, \underline{e}_2, \underline{e}_3]\,\underline{U}$ Then $Cov(\underline{X}) = \underline{P}\,\underline{\Lambda}\,\underline{P}^\phi = \underline{\Sigma}$

Hence simulate n observation vectors from $\underline{U}$ and let

$\underline{X} = \underline{P}\,\underline{U} + \underline{\mu}$

Problem 2

$H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$ $\quad H_1: \alpha_i \neq 0$ for minst en i, $i = 1, 2, 3$

$F_{obs} = 15.28.$ $\quad P(F_{2,6} \geq 15.28) = 0.004 < 0.05$. We conclude that at least one $\alpha_i \neq 0$, i.e. the declination has changed.

There are three pairwise comparisons $\mu_2 - \mu_1$, $\mu_3 - \mu_1$, $\mu_3 - \mu_2$

Only the confidence interval $\mu_3 - \mu_1$ does not cover 0 using the exact Tukey test. Hence we can only conclude that $\mu_3 \neq \mu_1$

The corresponding time span is $1805 - 1669 = 196$ year.

The two others time span are: 111 and 85 respectively, but according to the Tukey test we cannot conclude that 100 year is sufficient to conclude there has been a change although the comparison for $\mu_3 - \mu_2$ is pretty close.

b) $\beta_1 = \mu + \alpha_1$, $\beta_2 = \mu + \alpha_2$, $\beta_3 = \mu + \alpha_3$

$$\mathrm{Cov}(\hat{\beta}) = \sigma^2 (X^t X)^{-1} = \sigma^2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \sigma^2 \frac{1}{3} I$$

Hence $\hat{\beta}_1$, ~~and~~ $\hat{\beta}_2$ and $\hat{\beta}_3$ are uncorrelated. Since the Y_is are normal distributed they are also independent.

c)

$$H = X \cdot \frac{1}{3} I \cdot X^t = \frac{1}{3} X X^t = \frac{1}{3} \begin{bmatrix} J_3 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

Since $H = J^*$, it follows that expression 1 and 2 are equal.

All the matrices defined are shown in to be projection matrices in the course. Hence

$$\text{rank}\left(I - \frac{1}{9} J\right) = tr(I) - \frac{1}{9} tr(J) = 9 - \frac{1}{9} \cdot 9 = 8$$

$$\text{rank}(I - J^*) = \text{rank}(I - H) = tr(I) - tr(H) = 9 - \frac{1}{3} \cdot 9 = 6$$

$$\text{rank}\left(J^* - \frac{1}{9} J\right) = \text{rank}\left(H - \frac{1}{9} J\right) = tr(H) - \frac{1}{9} tr(J) = 3 - 1 = 2$$

$$\text{Rank}(H) = tr(H) = 3$$

d) According to our theoretical results since $(I - H)(Y - X\beta) = (I - H)Y$, and $Y \sim N(X\beta, \sigma^2 I)$, the two quadratic forms are independent if $(I - H) H = 0$. $(I - H) H = H - H^2 = H - H = 0$.

Hence they are independent.

Also: $\dfrac{(Y - X\beta)^t (I - H)(Y - X\beta)}{\sigma^2} = \dfrac{Y^t (I - H) Y}{\sigma^2}$ is $\chi^2(\text{rank}(I - H))$, i.e. $\chi^2(6)$

If $\beta_1 = \beta_2 = \beta_3 = 0$. $E(\underline{Y}) = 0$. The Y's are indepent and $Var(Y_{ij}) = \sigma^2$, $\forall i,j$. Hence $\frac{\underline{Y}^t H \underline{Y}}{\sigma^2} \sim \chi^2(\text{rank}(H)) \sim \chi^2(3)$

e)

H_0^1: $\mu_1 = \mu_2 = \mu_3$ $\quad$ H_1^1: $\mu_i \neq \mu_j$ for at least one i,j

This is the normal F-test in a one-way anova.

$$F = \frac{SS_H/2}{SS_E/6} = \frac{\underline{Y}^t(\bar{\underline{J}} - \frac{1}{9}\underline{J})\underline{Y}/2}{\underline{Y}^t(\underline{I} - \bar{\underline{J}})\underline{Y}/6} \sim F_{2,6}.$$

From theory we know that $\frac{SS_H}{\sigma^2}$ and $\frac{SS_E}{\sigma^2}$ are independent and $\chi^2(\text{rank}(\bar{\underline{J}} - \frac{1}{9}\underline{J}))$ and $\chi^2(\text{rank}(\underline{I} - \bar{\underline{J}}))$ respectively. Thereby the distribution. From 2a we know that H_0^1 will be rejected.

H_0^2 $\mu_1 = \mu_2 = \mu_3 = 0 \Leftrightarrow \beta_1 = \beta_2 = \beta_3 = 0$, H_1^2: at least one $\mu_i \neq 0$

From the discussion in 2c and 2d we know that under H_0^2: $\mu_1 = \mu_2 = \mu_3 = 0$

$$\frac{\underline{Y}^t H \underline{Y}/3}{\underline{Y}^t(\underline{I} - H)\underline{Y}/6} \sim F_{3,6}.$$ From the output $P(F_{3,6} \geq F_{obs}) \approx 5.482 \cdot e^{-10}$. Hence H_0^2 is rejected.

$SS_R = \underline{Y}^t(\underline{H} - \frac{1}{n}\bar{\underline{J}})\underline{Y} = \underline{y}^t(\bar{\underline{J}} - \frac{1}{9}\underline{J})\underline{y} = 90.03$ from R-output before 2a

$SS_T = SS_R + SS_E = 90.03 + 17.67 = 107.70$,

$R^2 = \frac{90.03}{107.70} \approx 0.84$