NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4267 Linear Statistical Models**

Academic contact during examination:

Phone:

Examination date: August 2024

Examination time (from–to):

Permitted examination support material: C:
– Tabeller og formler i statistikk, Tapir forlag,
– K.Rottman. Matematisk formelsamling,
– Stamped yellow A5 sheet with your own handwritten notes,
– Calculator: HP30S, Citizen SR-270X, Citizen SR-270X College or Casio fx-82ES PLUS.

Other information:
Sensur:

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Let $X_1, ..., X_n$ be independent univariate random variables with the same distribution function $F(x)$. Denote

$$X_{\min} = \min\{X_1, ..., X_n\},\ X_{\max} = \max\{X_1, ..., X_n\}.$$

a) Find the distribution function of the bivariate random vector $Y = (X_{\min}, X_{\max})^T$.

Problem 2 Let X be a trivariate random vector with zero expectation and the covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

with $0 < \rho < 1$. Suppose that X and a non-random vector $a = (a_1, a_2, a_3)^T$ satisfy condition

$$a^T X = c,$$

where c is a real number, $c \neq 0$.

a) Find ρ.

Problem 3 Let $X = (X_1, X_2)^T$ be a bivariate normal random vector with zero expectation and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

a) Find probability $P(X_1 > 2X_2)$.

b) Find the distribution function of X.

Problem 4 The normal multiple linear regression model can be written in matrix notation as

$$Y = X\beta + \epsilon,$$

where Y is an n-dimensional random column vector, X is a fixed design matrix with n rows and p columns, β is an unknown p-dimensional vector of regression parameters and ϵ is an n-dimensional column vector of multivariate normal random errors with mean $E(\epsilon) = 0$ and covariance matrix $\mathrm{Cov}(\epsilon) = \sigma^2 I$, where I is the $n \times n$ identity matrix. Assume that $n > p$ and that X has rank p. R input and output for this model are given below. Use information from this print-out.

```
Call:
lm(formula = Y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-1.3776 -0.5548 -0.1497  0.7190  1.2922

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.4977     0.5832  -0.853 0.405281
x1            1.3058     0.4517   2.891 0.010165 *
x2            2.1532     0.4517   4.766 0.000179 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8248 on 17 degrees of freedom
Multiple R-squared:  ?,         Adjusted R-squared:  0.5435
F-statistic: 12.31 on ? and ? DF,  p-value: 0.0004947
```

a) In the R output three numerical values are replaced by question marks. Calculate numerical values and provide details of the calculations.

b) The matrix X is such that

$$(X^TX)^{-1} = \begin{pmatrix} 0.5 & -0.3 & -0.3 \\ -0.3 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0.3 \end{pmatrix}.$$

Perform a test for

$$H_0 : \beta_1 + \beta_2 = 0$$

vs.

$$H_1 : \beta_1 + \beta_2 \neq 0.$$

The significance level is 0.05.

c) Three hypotheses

1) $H_0 : \beta_0 = 0$ vs. $H_1 : \beta_0 \neq 0$,

2) $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$,

3) $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 \neq 0$

are tested simulteneously. The family-wise error rate (FWER) is 0.01. Which null hypotheses are rejected if the Bonferroni method is used? Which null hypotheses are rejected if the Šidák method is used?

d) Perform a test for

$$H_0 : \sigma^2 = 1$$

vs.

$$H_1 : \sigma^2 > 1.$$

The significance level is 0.05.

Problem 5 Let Y have an exponential distribution.

a) Find an approximate variance-stabilizing transformation, i.e. a function $g(y)$ such that the variance of $g(Y)$ is approximately constant.

Problem 6 Let $Z = (Z_1, Z_2)^T$ be a bivariate standard normal vector.

a) Find probability

$$P(Z^T R Z \geq 161.45 Z^T S Z),$$

where

$$R = \begin{pmatrix} a & b \\ b & 1-a \end{pmatrix}, \; S = \begin{pmatrix} 1-a & b \\ b & a \end{pmatrix},$$

and $a^2 + b^2 = a$.