NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4267 Linear Statistical Models**

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Examination date: 30 May 2024

Examination time (from–to): 15:00 – 19:00

Permitted examination support material: C:
– Tabeller og formler i statistikk, Tapir forlag,
– K.Rottman. Matematisk formelsamling,
– Stamped yellow A5 sheet with your own handwritten notes,
– Calculator: HP30S, Citizen SR-270X, Citizen SR-270X College or Casio fx-82ES PLUS.

Other information:
Sensur:

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Let a univariate random variable X have the distribution function $F(x)$.

a) Find distribution functions of bivariate random vectors $Y = (X, X)^T$ and $Z = (X, e^X)^T$.

Problem 2 Let Z be a bivariate standard normal vector, and let A and B be matrices of the form

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix},$$

where a and b are real numbers.

a) Find a and b if AZ and BZ are independent.

Problem 3 A multiple linear regression model is considered:

$$Y = X\beta + \epsilon,$$

where X is an $n \times p$ design matrix, β is a p-dimensional parameter vector, ϵ is n-dimensional multinormal with zero mean and covariance matrix $\sigma^2 I$. R input and output for this model are given below. Use information from this print-out.

```
Call:
lm(formula = Y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-0.8819 -0.3581  0.1598  0.3356  1.0157

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.2119          ?  -0.518  0.61123
x1            1.2285     0.3170       ?  0.00121 **
x2                ?      0.3170   6.952 2.34e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.5787 on 17 degrees of freedom
Multiple R-squared:  0.7502,    Adjusted R-squared:  ?
F-statistic: 25.53 on 2 and 17 DF,  p-value: 7.576e-06
```

a) In the R output four numerical values are replaced by question marks. Calculate numerical values and provide details of the calculations. What is the relationship between the second and third diagonal elements of the matrix $(X^TX)^{-1}$? Justify the answer.

b) The following information about X is available: $x_{i1} = 1$ if $i \leq 3n/4$, $x_{i1} = 0$ if $i > 3n/4$, $x_{i2} = 0$ if $i \leq n/4$, $x_{i2} = 1$ if $i > n/4$. What is the correlation coefficient between $\hat{\beta}_1$ and $\hat{\beta}_2$? The following information (R input and output) might be useful:

```
> matrix<-array(c(20,15,15,15,15,10,15,10,15),dim = c(3,3))
> inverse<-solve(matrix)
> inverse
     [,1] [,2] [,3]
[1,]  0.5 -0.3 -0.3
[2,] -0.3  0.3  0.1
[3,] -0.3  0.1  0.3
```

c) Find 95% confidence interval for β_0.

d) Perform a test for

$$H_0 : \beta_1 = \beta_2$$

vs.

$$H_1 : \beta_1 \neq \beta_2.$$

The significance level is 0.05.

e) Three hypotheses

1) $H_0 : \beta_0 = 0$ vs. $H_1 : \beta_0 \neq 0$,

2) $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$,

3) $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 \neq 0$

are tested simulteneously. Which null hypotheses are rejected if the Šidák method is used to keep the family-wise error rate (FWER) below 0.05. Why?

f) Now suppose that $\sigma = 1$. What is the expectation and variance of the quadratic form $(Y - X\beta)^TX(X^TX)^{-1}X^T(Y - X\beta)$?

Problem 4 Let Y have a chi-square distribution with m degrees of freedom.

a) Find an approximate variance-stabilizing transformation, i.e. a function $g(y)$ such that the variance of $g(Y)$ is approximately constant.

Problem 5 Let a bivariate random vector $X = (X_1, X_2)^T$ have zero expectation and the covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

a) Prove that

$$E \max\{X_1^2, X_2^2\} \leq 1 + \sqrt{1 - \rho^2}.$$

Hint: Use the Cauchy-Schwarz inequality $E(UV) \leq \sqrt{EU^2}\sqrt{EV^2}$