Transformations

Let $\boldsymbol{Y} = g(\boldsymbol{X})$ one to one and let $\boldsymbol{X} = h(\boldsymbol{Y})$.

Then $f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(h(\boldsymbol{y}))|\boldsymbol{J}|$ where the ij element in the determinant $\boldsymbol{J}$ is $\frac{\partial x_i}{\partial y_j}$.

Multivariate moment generating function

$M_{\boldsymbol{X}}(\boldsymbol{t}) = E\left[e^{\boldsymbol{t}^T\boldsymbol{X}}\right] = M_{\boldsymbol{t}^T\boldsymbol{X}}(1)$.

Multivariate normal

Definition: $\mathbf{X}$ multivariate normal if $\boldsymbol{a}^T\boldsymbol{X}$ is univariate normal distributed $\forall \boldsymbol{a} \in R^p$.

$\mathbf{X}$ multivariate normal $\Rightarrow \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{c}$ is also multivariate normal.

Using the transformation $\boldsymbol{Z} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})$ where $\boldsymbol{Z} \sim \boldsymbol{N}_p(\mathbf{0}, \boldsymbol{I})$

We get: $f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2} e^{\frac{-(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}}$

Note: $\left|\boldsymbol{\Sigma}^{-1/2}\right| = \left|\boldsymbol{P}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{P}^T\right| = \left|\boldsymbol{P}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{P}^{-1}\right| = |\boldsymbol{P}|\left|\boldsymbol{\Lambda}^{-1/2}\right|\left|\boldsymbol{P}^{-1}\right|$

$= \left|\boldsymbol{\Lambda}^{-1/2}\right| = \prod_{i=1}^{p} \lambda_i^{-1/2} = \left(\prod_{i=1}^{p} \lambda_i\right)^{-1/2} = |\boldsymbol{\Sigma}|^{-1/2}$

The multivariate normal density is constant on ellipsoids i.e. $\forall \boldsymbol{x}$ such that $\left(\boldsymbol{x}-\boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)=c^2$. The axis are given by $c\lambda_i^{-1/2}\boldsymbol{e}_i$, i=1,2,…,p.