Independent linear transformations in the multivariate normal

$\boldsymbol{X}_p \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. $\boldsymbol{AX}$ and $\boldsymbol{BX}$ independent $\Leftrightarrow \boldsymbol{A\Sigma B}^T = \mathbf{0}$

Independence and conditional distributions for multivariate normal distributions.

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix},\ \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix},\ \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix},\ |\boldsymbol{\Sigma}_{22}| > 0.$$

a) $\boldsymbol{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{X}_2 - \boldsymbol{\mu}_2)$ and $(\boldsymbol{X}_2 - \boldsymbol{\mu}_2)$ are independent.

b) $\boldsymbol{X}_1 | \boldsymbol{X}_2 = \boldsymbol{x}_2 \sim \boldsymbol{N}_q\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)$

Expectation of quadratic forms

$\boldsymbol{X}_p$ random vector and $\boldsymbol{A}_{p \times p}$ a symmetric matrix.

$$E\left[\boldsymbol{X}^T \boldsymbol{AX}\right] = \boldsymbol{\mu}^T \boldsymbol{A\mu} + tr[\boldsymbol{A\Sigma}]$$

Distribution of quadratic forms

$\boldsymbol{X}_p \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $|\boldsymbol{\Sigma}| > 0$. Then $(\boldsymbol{X} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi^2(p)$

Can be used for assessing multivariate normal distributions for the data $\quad \hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i$.

Estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

p-dimensional random sample: $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i = \bar{\boldsymbol{X}},\ \boldsymbol{S} = \frac{1}{k}\sum_{i=1}^{n}\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)^T,\ k = n \text{ or } n\text{-}1$$

$\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ multivariate normal implies ML estimators $\hat{\boldsymbol{\mu}}$ and $\boldsymbol{S}$ with *k=n*. *k=n-1* implies unbiased.

Centering matrix

$$\boldsymbol{I}_{n\times n} - \frac{1}{n}\left(\mathbf{1}\mathbf{1}^T\right)_{n\times n} = \begin{bmatrix} 1-1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & \cdots & -1/n \\ \vdots & \vdots & \vdots & \vdots \\ -1/n & -1/n & \cdots & 1-1/n \end{bmatrix}$$

Idempotent matrices

A square matrix **A** is idempotent if **A A= A.**

If it is both idempotent and symmetric it is called a projection matrix.