Maximum likelihood estimation of regression parameters

$\boldsymbol{\varepsilon} : N\left(\mathbf{0},\sigma^2\boldsymbol{I}\right) \Rightarrow \hat{\boldsymbol{\beta}}_{ML} = \hat{\boldsymbol{\beta}}_{LS}$ and $\hat{\boldsymbol{\beta}}_{ML} \sim N\left(\boldsymbol{\beta}, \sigma^2\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)$

Partitioning of variation

$$\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2 = \sum_{i=1}^{n}\left(y_i-\hat{y}_i\right)^2 + \sum_{i=1}^{n}\left(\hat{y}_i-\bar{y}\right)^2 \text{ or}$$

$$SS_T = SS_E + SS_R \text{ or}$$

$$\boldsymbol{y}^T\left(\boldsymbol{I}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\boldsymbol{y} = \boldsymbol{y}^T\left(\boldsymbol{I}-\boldsymbol{H}\right)\boldsymbol{y} + \boldsymbol{y}^T\left(\boldsymbol{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\boldsymbol{y}$$

where $\boldsymbol{H}$, $\frac{1}{n}\mathbf{1}\mathbf{1}^T$, $\boldsymbol{I}-\boldsymbol{H}$, $\boldsymbol{I}-\frac{1}{n}\mathbf{1}\mathbf{1}^T$, $\boldsymbol{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T$ are idempotent.

Specific results in linear regression

R1 $E\left[\boldsymbol{\varepsilon}\right]=\mathbf{0}, Cov\left[\boldsymbol{\varepsilon}\right]=\sigma^2\boldsymbol{I} \Rightarrow E\left[\hat{\sigma}^2_{LS}\right] = E\left[\sum_{i=1}^{n}\frac{\left(Y_i-\hat{Y}_i\right)^2}{n-\left(k+1\right)}\right]=\sigma^2$

R2 $\boldsymbol{\varepsilon} \sim N\left(\mathbf{0},\sigma^2\boldsymbol{I}\right) \Rightarrow \frac{\boldsymbol{Y}^T\left(\boldsymbol{I}-\boldsymbol{H}\right)\boldsymbol{Y}}{\sigma^2} \sim \chi^2\left(n-k-1\right)$

R3a) $\left.\begin{array}{c}\varepsilon \sim N\left(\mathbf{0},\sigma^2\boldsymbol{I}\right) \\ \beta_1=\beta_2=\cdots=\beta_k=0\end{array}\right\} \Rightarrow \frac{\boldsymbol{Y}^T\left(\boldsymbol{I}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\boldsymbol{Y}}{\sigma^2} \sim \chi^2\left(n-1\right)$

b) $$\left.\begin{array}{c}\varepsilon \sim N\left(\mathbf{0},\sigma^2\boldsymbol{I}\right)\\ \beta_1=\beta_2=\cdots=\beta_k=0\end{array}\right\}\Rightarrow \frac{\boldsymbol{Y}^T\left(\boldsymbol{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\boldsymbol{Y}}{\sigma^2}\sim\chi^2\left(k\right)$$

R4 $\boldsymbol{\varepsilon}\sim N\left(\mathbf{0},\sigma^2\boldsymbol{I}\right)\Rightarrow\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ are independent

Fisher distribution

$$X\sim\chi^2\left(\nu_1\right),\ Y\sim\chi^2\left(\nu_2\right)\Rightarrow\frac{X/\nu_1}{Y/\nu_2}\sim F_{\nu_1,\nu_2}$$