Some rules, providing the expectations exist

$E[\alpha \underline{X} + \beta \underline{Y}] = \alpha E[\underline{X}] + \beta E[\underline{Y}]$ linearity.

$\underline{A}_{(q \times p)}$: $E[\underline{A}\,\underline{X}] = \underline{A}\, E[\underline{X}]$

The i-th element in $E[\underline{A}\,\underline{X}]$ is $E\left[\sum_{k=1}^{p} a_{ik} X_k\right] = \sum_{k=1}^{p} a_{ik} E[X_k]$

where a_{ik} is the element on place ik in $\underline{A}$

In general: Let $\underline{X}$ be a matrix with random variable X_{ij} on place ij. Let $\underline{A}$ and $\underline{B}$ be matrices such that $\underline{A}\,\underline{X}\,\underline{B}$ is defined. Then $E[\underline{A}\,\underline{X}\,\underline{B}] = \underline{A}\, E[\underline{X}]\, \underline{B}$

Let $\underline{X}$ and $\underline{Y}$ be independent random vectors. Then

$$E[\underline{X}\,\underline{Y}^T] = E\left[\begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix} (Y_1, \ldots, Y_q)\right]_{p \times q} = (E\underline{X})(E\underline{Y})^T$$

~~alle komponenter i $\underline{X}$ er uavhengige av komponenter i $\underline{Y}$~~

The Covariance matrix $\underline{\Sigma} = Var[\underline{X}]$

Univariate: $Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$

$Y = X$: $Cov[X,X] = E[(X - E[X])^2] = Var[X]$

Let $\underline{X}$ be a random vector, $\underline{X} = [X_1, \ldots, X_p]$.

$$Cov[\underline{X}] \stackrel{\Delta}{=} E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T]_{p \times p} = Var[\underline{X}] = \underline{\Sigma}$$

$$= E[\underline{X}\,\underline{X}^T] - E[\underline{X}\, E[\underline{X}]^T] - E[E[\underline{X}]\,\underline{X}^T] + E[E[\underline{X}] \cdot E[\underline{X}]^T]$$

$$= E[\underline{X}\,\underline{X}^T] - E[\underline{X}] \cdot E[\underline{X}]^T$$

Also.

$$Cov[\underline{X}] = E\left[\begin{pmatrix} X_1 - E[X_1] \\ \vdots \\ X_p - E[X_p] \end{pmatrix}(X_1 - E[X_1], \dots, X_p - E[X_p])\right]_{p \times p}$$

has $Cov[X_i, X_j]$ on place ij and the variances to each X_i on the diagonal.

Example

$$f(x,y) = \begin{cases} 6xy^2, & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = 2x,\quad 0<x<1$$

$$f_Y(y) = 3y^2,\quad 0<y<1$$

$$E[X] = \int_{-\infty}^{\infty} x\, 2x\, dx = \int_0^1 2x^2 dx = \left[2\frac{x^3}{3}\right]_0^1 = \frac{2}{3} \qquad E[Y] = \int_0^1 3y^3 dy = \frac{3}{4}$$

$$E\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2/3 \\ 3/4 \end{bmatrix}$$

$$Var[X] = \int_0^1 2x^3 dx - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$Var[Y] = \int_0^1 3y^4 dy - (E[Y])^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

$$Cov\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1/18 & 0 \\ 0 & 3/80 \end{bmatrix}$$ since X and Y are independent

5. $Cov[\underline{X}]$ is symmetric $(Cov(X_i, X_j) = Cov(X_j, X_i))$
and positive semidefinite i.e. ~~$\underline{a}^T Cov[\underline{X}]\, \underline{a}$~~ $\underset{1\times p}{\underline{a}^T}\, \underset{p\times p}{\Sigma}\, \underset{p\times 1}{\underline{a}} \geq 0$
for all vectors $\underline{a}$

If $\underset{\sim}{X}_{p\times 1}$ and $\underset{\sim}{Y}_{q\times 1}$ are two random vectors

we have $Cov[\underline{X}, \underline{Y}]_{p\times q} = E[(\underline{X}-E[\underline{X}])(\underline{Y}-E[\underline{Y}])^T]_{p\times q}$

$= E[\underline{X}\,\underline{Y}^T] - E[\underline{X}]E[\underline{Y}]^T$

We have $Cov[X_i, Y_j]$ on place ij. If $\underline{X}$ and $\underline{Y}$ are independent $Cov[\underline{X}, \underline{Y}]$ is a $\underline{0}_{p\times q}$ matrix

Properties

1. $Cov[\underline{A}\,\underline{X} + \underline{b}] = \underline{A}(Cov[\underline{X}])\underline{A}^T$

Proof. $\underline{V} = \underline{A}\,\underline{X} + \underline{b} \Rightarrow \underline{V} - E[\underline{V}] = \underline{A}\,\underline{X} + \underline{b} - \underline{A}E[\underline{X}] - \underline{b} = \underline{A}(\underline{X} - E[\underline{X}])$

$(\underline{V} - E[\underline{V}])^T = (\underline{X} - E[\underline{X}])^T\underline{A}^T$

Hence $Cov[\underline{A}\,\underline{X} + \underline{b}] = \underline{A}E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T]\underline{A}^T = \underline{A}\,Cov(\underline{X})\underline{A}^T$

2. $Cov[(\underline{X}+\underline{Y}), \underline{Z}] = E[(\underline{X} + \underline{Y} - E[\underline{X}+\underline{Y}])(\underline{Z} - E[\underline{Z}])^T]$

$= E[(\underline{X} - E[\underline{X}])(\underline{Z} - E[\underline{Z}])^T] + E[(\underline{Y} - E[\underline{Y}])(\underline{Z} - E[\underline{Z}])^T]$

$= Cov[\underline{X}, \underline{Z}] + Cov[\underline{Y}, \underline{Z}]$. Univariate $Cov[X+Y, Z] = Cov(X,Z) + Cov(Y,Z)$

3. $Cov[\underline{X} + \underline{Y}] = E[(\underline{X} + \underline{Y} - E[\underline{X}+\underline{Y}])(\underline{X} + \underline{Y} - E[\underline{X}+\underline{Y}])^T]$

$= E[(\underline{X} - E[\underline{X}] + \underline{Y} - E[\underline{Y}])(\underline{X} - E[\underline{X}] + \underline{Y} - E[\underline{Y}])^T]$

$= Cov[\underline{X}] + Cov(\underline{Y}, \underline{X}) + Cov(\underline{X}, \underline{Y}) + Cov(\underline{Y})$

univariate: $Var[X+Y] = Var[X] + 2\,cov(X, Y) + Var(Y)$

4. $Cov(\underline{A}\underline{X}, \underline{B}\underline{Y}) = E[(\underline{A}(\underline{X} - E(\underline{X})))(\underline{Y} - E[\underline{Y}])^T \underline{B}^T)]$

$= \underline{A}\ Cov(\underline{X}, \underline{Y})\underline{B}^T$

Univariate ~~Cov(ax~~ $Cov(aX, bY) = ab\ Cov(X, Y)$

Why is $Cov(\underline{X})$ positive semidefinite?

Let $\underline{a}$ be a $p \times 1$- vector. $\underline{a}^T Cov(\underline{X})\underline{a}$ is a quadratic form
symmetric

Then $\underline{a}^T(Cov[\underline{X}])\underline{a} = Cov(\underbrace{\underline{a}^T \underline{X}}_{\text{skalar.}}) = Var(\underline{a}^T\underline{X}) \geq 0$

If. $\underline{a}^T(cov[\underline{X}])\underline{a} = 0 \Rightarrow Var(\underline{a}^T\underline{X}) = 0 \Rightarrow \underline{a}^T\underline{X}$ essentially is a constant.

<u>Spectral decomposition of symmetric matrices.</u>

Let $\underline{X}$ be a $p \times 1$ random vector and $\underline{\Sigma} = Cov(\underline{X})$

$\underline{\Sigma}$ is symmetric and positive semidefinite.

$\underline{\Sigma}$ symmetric implies that $\underline{\Sigma}$ can be expressed in terms of. p eigenvalue-eigenvectors pairs (λ_i, e_i)

orthogonal $(\underline{e}_i^T \underline{e}_j = 0)$

eigenvalue eigenvectors with length.

as = $\underline{\Sigma} = \sum_{i=1}^{p} \lambda_i \underline{e}_i \underline{e}_i^T = \underbrace{\begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_p \\ & & & \\ & & & \end{bmatrix}}_{\underline{P}} \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_p \end{bmatrix}}_{\underline{\Lambda}} \underbrace{\begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_p^T \end{bmatrix}}_{P^T}$

$\underline{a}^T \Sigma \underline{a} > 0, \forall \underline{a} \neq \underline{0}$

$= \underline{P}\underline{\Lambda}\underline{P}^T, \quad \underline{P}\underline{P}^T = \underline{P}^T\underline{P} = \underline{I}$

$\underset{\sim}{\Sigma}$ positive definite i.e $\underline{a}^T \underset{\sim}{\Sigma} \underline{a} > 0, \; \forall \, \underline{a} \neq \underline{0}$

$\Longleftrightarrow \; \lambda_i > 0, \; i = 1, 2, \ldots, p$. Then $\underset{\sim}{\Sigma}^{-1}$ exists.