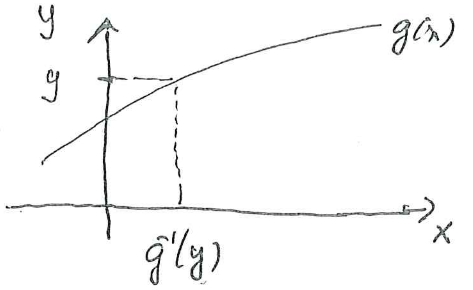
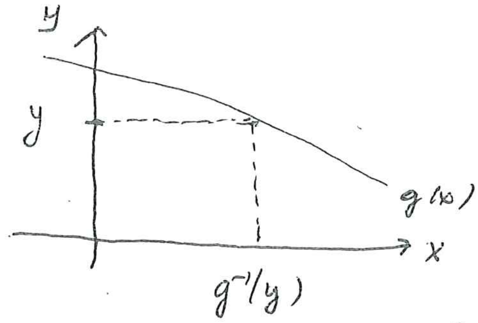
Transformations

Let X be a continuous random variable with pdf given by $f_X(x)$ and let $Y = g(X)$ where $g(x)$ is a one to one differentiable function. Then $g(x)$ is either monotonically increasing or decreasing



$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

$$= \begin{cases} P(X \le g^{-1}(y)), & g \text{ monoton increasing} \\ P(X \ge g^{-1}(y)), & g \text{ monoton decreasing} \end{cases}$$



$$= \begin{cases} F_X(g^{-1}(y)), & g \text{ monoton increasing} \\ 1 - F_X(g^{-1}(y)), & g \text{ monoton decreasing} \end{cases}$$

This implies

$$f_Y(y) = f_X(g^{-1}(y))\,\frac{dx}{dy}, \quad g \text{ monoton increasing}$$

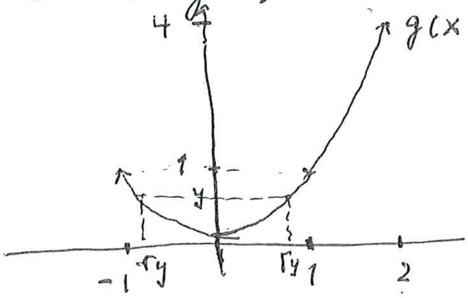
$$f_Y(y) = -f_X(g^{-1}(y))\,\frac{dx}{dy}, \quad g \text{ monoton decreasing}$$

But in the last case $\frac{dx}{dy}$ is negative $\Rightarrow$ we can write

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| \quad \text{always}.$$

__Example.__ $g(x)$ not monoton. Suppose X has support $-1 < x < 2$

Let $Y = g(X) = X^2$. Y has support $[0, 4)$



For $y \in [1, 4)$ we get $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$= P(X \le \sqrt{y}) = F_X(\sqrt{y})$ and $f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$

For $0 \le y < 1$, $F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \Rightarrow f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}}$

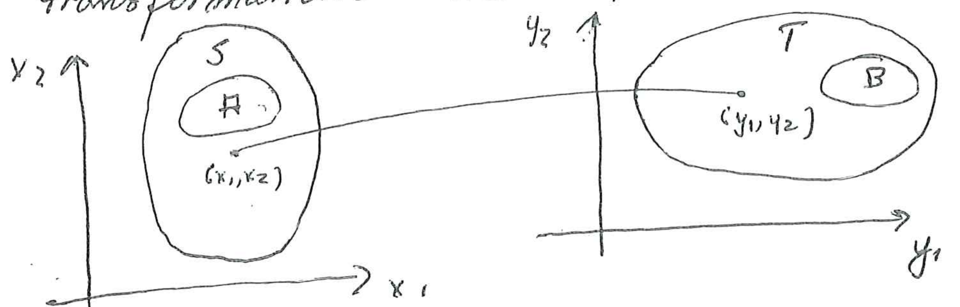
$$= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

Continuous multivariate case

Bivariate

Let $\underline{X} = (X_1, X_2)^T$ have a joint pdf $f_{X_1,X_2}(x_1, x_2)$
with support set S
Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$

where $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformations that maps S in R^2 onto T in R^2



$$P((Y_1, Y_2) \in B) = P((X_1, X_2) \in A) = \iint_A f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

Let $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$. Mathematically it is shown that,

$$P((Y_1, Y_2) \in B) = \iint_A f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 = \iint_B f_{X_1,X_2}(w_1(y_1,y_2), w_2(y_1,y_2))\,|J|\, dy_1\, dy_2$$

where $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$ is the Jacobian.

General: $\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_p} \\ \vdots & & \\ \frac{\partial x_p}{\partial y_1} & \cdots & \frac{\partial x_p}{\partial y_p} \end{vmatrix}$

Example. $X, Y \sim N(0,1)$ and independent

Let $U = X + Y$ and $V = X - Y$. Find pdf to $[U, V]^T$

$$\text{Pdf of } [X, Y]^T = f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$\left.\begin{array}{l} u = x + y \\ v = x - y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{array}\right. \Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \text{ and } |J| = \frac{1}{2}$$

$$\text{Pdf } [U, V]^T = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right)} \cdot \frac{1}{2}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{u^2}{2}\right)} \cdot \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{1}{2}\left(\frac{v^2}{2}\right)}$$

which is the joint pdf of two independent and $N(0,2)$ variables.

Assume $\underline{u} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$ linear i.e $\underline{Y} = \underline{U}(\underline{X}) = \underline{A}\,\underline{X} + \underline{b}$

and $\underline{X} = \underline{A}^{-1}(\underline{Y} - \underline{b})$. $J(\underline{y}) = \det \underline{A}^{-1} = \frac{1}{\det \underline{A}}$

Pdf $\underline{Y}$: $f_{\underline{X}}(\underline{A}^{-1}(\underline{y} - \underline{b})) \frac{1}{|\det \underline{A}|}$

Multivariate momentgenerating function

Univariate $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$, X continuous.

If it exist and derivatives exist on an interval containing 0, it can be used to find moments.

$M_X'(0) = E[X]$, $M_X''(0) = E[X^2]$, ...

If for two variables X and Y, $M_X(t) = M_Y(t)$, they have the same probability distribution.

Also, $X_1, \ldots, X_p$ independent $\Rightarrow M_{\sum_{i=1}^{p} X_i}(t) = \prod_{i=1}^{p} M_{X_i}(t)$.

Note: $M_{X^2}(t) = E[e^{tX^2}]$

Multivariate momentgenerating function

Let $\underline{X}$ be a random vector. $M_{\underline{X}}(\underline{t}) \triangleq E[e^{\underline{t}^T \underline{X}}] = E[e^{\sum_{i=1}^{p} t_i X_i}]$

$\underline{t}$ has the same dimension as $\underline{X}$.

Let $\underline{X}$ and $\underline{Y}$ be two random vectors. If $M_{\underline{X}}(\underline{t}) = M_{\underline{Y}}(\underline{t})$ and exist in a surrounding of $\underline{0}$, then $\underline{X}$ and $\underline{Y}$ have the same multivariate distribution. [HS uses characteristic function $E[e^{i\underline{t}\underline{X}}]$ instead].

Note, $e^{\underline{t}^T \underline{X}}$ is a scalar. hence $M_{\underline{X}}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\underline{t}^T \underline{x}} f(\underline{x})\, d\underline{x}$

Let $\underline{X} = [X_1, \ldots, X_p]^T$ and assume $X_1, \ldots, X_p$ are independent.

Then $M_{\underline{X}}(\underline{t}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\sum_{i=1}^{p} t_i x_i} f(\underline{x})\, d\underline{x} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{p} e^{t_i x_i}$ [redacted] $\prod_{i=1}^{p} f_i(x_i)\, d\underline{x}$

$= \prod_{i=1}^{p} \int_{-\infty}^{\infty} e^{t_i x_i} f_i(x_i)\, dx_i = \prod_{i=1}^{p} M_{X_i}(t_i)$

Example. $Z_1, \ldots, Z_p \overset{\text{each}}{\sim} N(0,1)$, $M_{Z_i} = e^{\frac{t^2}{2}} \Rightarrow M_{\underline{Z}}(\underline{t}) = \prod_{i=1}^{p} e^{t_i^2/2}$

where $\underline{Z} = [Z_1, \ldots, Z_p]$

$M_{\underline{Z}}(\underline{t}) = e^{\frac{1}{2}\sum_{i=1}^{p} t_i^2} = e^{\frac{1}{2}\underline{t}^T \underline{t}}$