A square matrix A that satisfies $AA = A$ is called idempotent.

Example

$A = \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix}$ $\quad AA = \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix}$

If A is idempotent and symmetric it is said to be a projection matrix $\left[\underline{I} - \frac{1}{n}\underline{1}\underline{1}^T\right]$ is a projection matrix.

Rank of symmetric idempotent matrices

RM1 Let $\underline{A}$ an $\underline{B}$ be two matrices such that their product $\underline{AB}$ is defined.

Then $rank(\underline{AB}) \leq \min(rank(\underline{A}), rank(\underline{B}))$

This follows since the columns of $\underline{AB}$ are linear combinations of the columns of $\underline{A}$ and the rows in $\underline{AB}$ are linear combinations of the rows of $\underline{B}$.

Example

$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\underline{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$\underline{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$b_{11}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_{21}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \qquad b_{12}\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_{22}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

We can do similar for row vectors.

Hence the column rank needs to be less or equal the rank of $\underline{A}$ and the row rank less or equal the rank of $\underline{B}$

<u>RM2</u>. $\underline{A}_{m\times m}$ symmetric implies rank($\underline{A}$) equals the number of nonzero eigenvalues.

<u>Proof</u>. $\text{rank}(\underline{A}) = \text{rank}(\underline{P}\underline{\Lambda}\underline{P}^T) \le \min(\text{rank}(\underline{P}), \text{rank}(\underline{\Lambda}\underline{P}^T))$

$\le \text{rank}(\underline{\Lambda}\underline{P}^T) = \text{rank}(\underline{P}^T\underline{P}\underline{\Lambda}\underline{P}^T) \le \min(\text{rank}(\underline{P}^T), \text{rank}(\underline{P}\underline{\Lambda}\underline{P}^T))$

$\le \text{rank}(\underline{P}\underline{\Lambda}\underline{P}^T) = \text{rank}(\underline{A})$

$\text{rank}(\underline{\Lambda}\underline{P}^T) \le \min(\text{rank}(\underline{\Lambda}), \text{rank}(\underline{P}^T)) \le \text{rank}(\underline{\Lambda})$

$= \text{rank}(\underline{\Lambda}\underline{P}^T\underline{P}) \le \text{rank}\ \underline{\Lambda}\underline{P}^T$

Hence $\text{rank}(\underline{A}) = \text{rank}(\underline{\Lambda}\underline{P}^T) = \text{rank}(\underline{\Lambda})$

<u>RM3</u>

$\underline{A}_{m\times m}$ symmetric and idempotent with rank r implies r eigenvalues are 1 and $m-r$ are zero

<u>Proof</u>. $\underline{A}\underline{x} = \lambda\underline{x} \Rightarrow \lambda\underline{x}^T\underline{x} = \underline{x}^T\underline{A}\underline{x} = \underline{x}^T\underline{A}\underline{A}\underline{x}$

$= \lambda\underline{x}^T\lambda\underline{x} = \lambda^2\underline{x}^T\underline{x}$

Hence $\underline{x}^T\underline{x}\,\lambda(\lambda-1) = 0 \Rightarrow \lambda = 0$ or $\lambda = 1$

Since the number of nonzero eigenvalues are r, r eigenvalues are 1 and $m-r$ are 0.

RM4 $\underline{A}_{m\times m}$ symmetric and idempotent implies

$tr(\underline{A}) = rank(\underline{A})$.

Proof. $tr(\underline{A}) = tr(\underline{P}\underline{\Lambda}\underline{P}^T) = tr(\underline{P}^T\underline{P}\underline{\Lambda}) = tr(\underline{\Lambda}) = rank(\underline{A})$

Example

$$\underline{X} = [\underline{X}_1, \dots, \underline{X}_m]^T = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1p} \\ \vdots & \vdots & & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mp} \end{bmatrix}$$

$$\underline{S} = \frac{\underline{X}^T(\underline{I} - \frac{1}{m}\underline{1}\underline{1}^T)\underline{X}}{m-1}$$

Then $rank(\underline{I} - \frac{1}{m}\underline{1}\underline{1}^T) = rank(\underline{I}) - \underbrace{rank(\frac{1}{m}\underline{1}\underline{1}^T)}_{= tr(\frac{1}{m}\underline{1}\underline{1}^T)} = m-1$

$(\underline{I} - \frac{1}{m}\underline{1}\underline{1})$ is called a centring matrix

RM5 Let $\underline{A}$ and $\underline{B}$ be symmetric and idempotent matrices such that $\underline{A}\underline{B} = 0$. Assume further that $X_1, X_2, \dots, X_m$ are independent normally distributed with expectation $\mu_1, \mu_2, \dots, \mu_m$ respectively and equal variance σ^2.

Let $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}$ and $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix}$. Then $(\underline{X} - \underline{\mu})^T\underline{A}(\underline{X} - \underline{\mu})$

and $(\underline{X}-\underline{\mu})^T\underline{B}(\underline{X}-\underline{\mu})$ are independent.

Proof. Let $\underline{U} = \underline{X} - \underline{\mu}$. Then $E[\underline{U}] = \underline{0}$ and $Cov[\underline{U}] = \sigma^2\underline{I}$

Further $\underline{U}^T\underline{A}\underline{U} = \underline{U}^T\underline{A}\underline{A}\underline{U} = \underline{Z}^T\underline{Z}$ where $\underline{Z} = \underline{A}\underline{U}$

and $\underline{U}^T\underline{B}\underline{U} = \underline{U}^T\underline{B}\underline{B}\underline{U} = \underline{V}^T\underline{V}$ where $\underline{V} = \underline{B}\underline{U}$

Z and V are independent if $E[\underline{Z}\underline{V}^T] = 0$

$$E[\underline{Z}\,\underline{V}^T] = E[\underline{A}\,\underline{U}\,\underline{U}^T\underline{B}] = \underline{A}\,E[\underline{U}\,\underline{U}^T]\underline{B} = \underline{A}\,\sigma^2\underline{I}\,\underline{B} = \sigma^2\underline{A}\underline{B} = \underline{0}$$

Distribution of Quadratic forms (extended)

RM 6. Let $\underline{A}_{m\times m}$ be a symmetric and idempotent matrix of rank r and let $Y_1, Y_2, \ldots, Y_m$ be independent normally distributed with expectation $\mu_1, \mu_2, \ldots, \mu_m$, respectively and equal variance σ^2. Define $\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$ and $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}$

Then $\dfrac{(\underline{Y}-\underline{\mu})^T\underline{A}(\underline{Y}-\underline{\mu})}{\sigma^2}$ is $\chi^2(r)$

Proof. $\underline{A} = \underline{P}\,\underline{\Lambda}\,\underline{P}^T$. Let $\underline{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}$ where $U_i = \dfrac{Y_i - \mu_i}{\sigma}$, $i = 1, 2, \ldots, m$

$U_1, U_2, \ldots, U_m$ are independent normally distributed random variables. $E[U_i] = 0$ and $SD[U_i] = 1$, $i = 1, 2, \ldots, m$.

We get $\dfrac{(\underline{Y}-\underline{\mu})^T\underline{A}(\underline{Y}-\underline{\mu})}{\sigma^2} = \underline{U}^T\underline{A}\,\underline{U} = \underline{U}^T\underline{P}\,\underline{\Lambda}\,\underline{P}^T\underline{U} = \underline{Z}^T\underline{\Lambda}\,\underline{Z} = \sum_{i=1}^{r} Z_i^2$

where $\underline{Z} = \underline{P}^T\underline{U}$. Further $Cov[\underline{Z}] = Cov[\underline{P}^T\underline{U}] = E[\underline{P}^T\underline{U}\,\underline{U}^T\underline{P}]$ $= \underline{P}^T E[\underline{U}\,\underline{U}^T]\underline{P} = \underline{P}^T\underline{I}\,\underline{P} = \underline{I}$ and $E[\underline{Z}] = E[\underline{P}^T\underline{U}] = \underline{P}^T E[\underline{U}] = \underline{0}$

Hence $Z_i \sim N(0,1)$, $i = 1, 2, \ldots, m$ and independent which implies that $\sum_{i=1}^{r} Z_i^2 \sim \chi^2(r)$

Example. Let the random sample $X_1, \ldots, X_n$ be univariate $N(\mu, \sigma^2)$.

Define $\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$. Then $\frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T \underset{\sim}{X} = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} \bar{X} \\ \bar{X} \\ \vdots \\ \bar{X} \end{bmatrix}$

and $\underset{\sim}{X} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T \underset{\sim}{X} = \left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{X} = \begin{bmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{bmatrix}$

and $S^2 = \dfrac{\underset{\sim}{X}^T \left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{X}}{n-1}$

$\left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right)(\underset{\sim}{X} - \underset{\sim}{\mu}) = \left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{X} - \left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{\mu}$, and $\underbrace{\left(\underset{\sim}{I} - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{\mu}}$ $= \underset{\sim}{\mu} - \underset{\sim}{\mu} = \underset{\sim}{0}$

since $\underset{\sim}{\mu} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}$

Hence $\dfrac{(n-1)}{\sigma^2} S^2 \sim \chi^2(n-1)$