Ex. 1 Age and memory

2-factor anova with interaction

Response Y : words recalled (out of 27)

Factor α : Age, 2 levels

Factor γ : Process, 5 levels

Model

mean recall age group j, processing method k

$$Y_{ijk} = \mu_{jk} + \varepsilon_{ijk} \qquad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

$$= \underbrace{\mu}_{\text{global mean}} + \underbrace{\alpha_j}_{\text{effect of age } j} + \underbrace{\gamma_k}_{\text{effect of process } k} + \underbrace{(\alpha\gamma)_{jk}}_{\text{added effect of combination } jk} + \varepsilon_{ijk}$$

$$\sum_j \alpha_j = 0 \qquad \sum_k \gamma_k = 0 \qquad \sum_j (\alpha\gamma_{jk}) = 0 \qquad \sum_k (\alpha\gamma_{jk}) = 0$$

R: table(words) hist(words)

NB: Y is a count $[0, 27]$

probability that subject in age grp. j and process k recalls a word

$$Y_{ijk} \sim \text{Binomial}(27, p_{jk})$$

$$E(Y_{ijk}) = 27 p_{jk} = \mu_{jk}$$

Normal approximation to discrete distn

From basic course, ok when $p \approx 0.5$

R: boxplot(words~Age)

- larger variation among younger?
- better recall younger?

this was already known before the study

R: boxplot(words ~ Process)

counting, rhyming (least active) → poor recall
reorder → increasing (as expected)

R: boxplot(words ~ Age * Process)
aggregate(....) ← averages in each group.

Interaction?
As in earlier plot, similarities for counting rhyming, larger differences for more 'active' processes.

NB: $\varepsilon_{ijk} \sim N(0, \sigma^2)$ Issues? Maybe.

R: fit model and look at residuals
→ Seem centered around 0, somewhat lower variance for inactive processes, but difficult to conclude with only 10 observations per group.

Tests: 1. Interaction?

↙ Yes
Stop?

↘ No
test main effect of age and process.

Recall: $\beta = \begin{bmatrix} \mu \\ \alpha_1 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ (\alpha\gamma)_{11} \\ (\alpha\gamma)_{12} \\ (\alpha\gamma)_{13} \\ (\alpha\gamma)_{14} \end{bmatrix}$ with $\beta_1 = \alpha_1$, $\vec{\beta}_2 = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, $\vec{\beta}_3 = ((\alpha\gamma)_{11}, (\alpha\gamma)_{12}, (\alpha\gamma)_{13}, (\alpha\gamma)_{14})$

$\alpha_2 = -\alpha_1$ ← younger (alphabetical)

$\gamma_5 = -\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4$

intentional (since we re-ordered in R)

$(\alpha\gamma)_{12} = -(\alpha\gamma)_{11}$ etc.

$H_0: \vec{\beta}_3 = 0$ vs $H_1: \vec{\beta}_3 \neq 0$ test interaction

R: Anova(model)

$p = 0.00027$ interaction is significant.

R: Summary(model)

(A) Find $\hat{\mu}_{11} = \hat{\mu} + \hat{\alpha}_1 + \hat{\gamma}_1 + (\hat{\alpha\gamma})_{11}$

$\hat{\mu}_{21} = \hat{\mu} + \hat{\alpha}_2 + \hat{\gamma}_1 + (\hat{\alpha\gamma})_{21}$

old is negative ← counting is a very bad strategy

$\hat{\mu}_{11} = 11.61 - 1.55 - 4.86 + \underbrace{1.8}_{\text{'add' back the age effect}} = 7$ words

$\hat{\mu}_{21} = 11.61 + 1.55 - 4.86 - 1.8 = 6.5$ words

$\boxed{\text{R: compare to mean, aggregate}}$

(B) Find $\hat{\mu}_{14} = \hat{\mu} + \hat{\alpha}_1 + \hat{\gamma}_4 + \widehat{(\alpha\gamma)}_{14}$

$\hat{\mu}_{24} = \hat{\mu} + \hat{\alpha}_2 + \hat{\gamma}_4 + \widehat{(\alpha\gamma)}_{24}$

← age is neg. ← imagery good. ← perhaps even worse to be old ...

$$\hat{\mu}_{14} = 11.61 - 1.55 + 3.89 - 0.55$$

$$= 13.4 \text{ words}$$

$$\hat{\mu}_{24} = 11.61 + 1.55 + 3.89 + 0.55 = 17.6 \text{ words}$$

Note: these predictions we also get from group means, but perhaps additional insight from model?

- older age negative, but only for 'active' processes.
  ↳ From book: perhaps they just don't try hard enough.

Discuss: Figure vs. model summary ...

Exam 2018, problem 3

response y
factor β, 4 levels

15×4 observations

$$Y_{ij} = \beta_j + \varepsilon_{ij} \qquad \varepsilon_{ij} \sim N(0, \sigma^2) \qquad j = 1, \dots, 4$$

↑
note no grand mean,
each β_j is the level mean.

a)

Aim $Y = X\beta + \varepsilon$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

← β_0, but we count factor levels from 1, and no intercept (β_0) in this model.

For X, assume now 2 observations (not 15)

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

NB third method for coding X!

$$(X^TX) \overset{\text{here}}{=} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad (X^TX)^{-1} = \frac{1}{2} I$$

in exercise, replace 2 with 15.

b) $\hat{\beta}_3 = 1.0902858$

$\hat{\beta}_4 = 0.1752633$

$SSE = 43.04524$

$H_0: \beta_3 = \beta_4 \qquad H_1: \beta_3 \neq \beta_4$

Recall:

$$F = \frac{(\underbrace{C}_{r \times p\text{-matrix}}\hat{\beta})^T (C(X^TX)^{-1}C^T)^{-1}(C\hat{\beta}) / \underbrace{r}_{\text{restrictions}}}{SSE/(n-\underbrace{p}_{\text{length of } \vec{\beta}})}$$

Here: $\beta_3 = \beta_4 \rightarrow \beta_3 - \beta_4 = 0$

$\rightarrow C\beta = [0 \;\; 0 \;\; 1 \;\; -1]\beta = 0 \qquad r = 1.$

$p = 4$

$n - p = 56$

$$C\hat{\beta} = \hat{\beta}_3 - \hat{\beta}_4 = 0.9150225$$

$$C(X^TX)C^T = \frac{1}{15}[0 \; 0 \; -1 \; 1] \, I \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{2}{15}$$

$$F = \frac{(0.9150225)^2 \cdot \frac{15}{2} / 1}{43.04524/56} = 8.169$$

critical value F-dist, $F_{1,56}$, $\alpha = 0.05$

$\approx 4 \rightarrow$ reject H_0, $\Rightarrow \beta_3 \neq \beta_4$

c) Bonferroni $m = 6$ tests

$$\alpha_{loc} = \frac{0.05}{6} = 0.0083$$

Reject $\beta_2 = \beta_3$ and $\beta_3 = \beta_4$.

Exam 2010

Response y : blood pressure reduction

Factor α : treatment, 3 levels

one level is "no intervention"

$$Y_{ij} = \mu_j + \varepsilon_{ij}$$

Test : (*) $\begin{cases} H_0: \mu_H = \mu_{No} \\ H_0: \mu_M = \mu_{No} \end{cases}$ comparing interventions to a baseline → use dummy coding

(**) $H_0 : \mu_H = \mu_M$

(*) Dummy coding :

$$\beta = \begin{bmatrix} \mu_{no} \\ \mu_H - \mu_{no} \\ \mu_m - \mu_{no} \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots \\ \vdots & \vdots & 0 \\ 1 & 0 & 1 \\ 1 & 0 & \vdots \\ 1 & 0 & 1 \end{bmatrix}$$

$$\beta_1 - \beta_2 = \mu_H - \mu_{no} - (\mu_m - \mu_{no}) = \mu_H - \mu_m$$

Test $\beta_1 = \beta_2$ $\qquad C\beta = 0$

$$C = [0 \quad 1 \quad -1]$$