Example 1: 2^3 full factorial (2 runs)

Model:

$$y = \beta_0 + \beta_1 \overset{T}{X_1} + \beta_2 \overset{C}{X_2} + \beta_3 \overset{pH}{X_3}$$

$$+ \beta_{12} X_1 X_2 + \beta_{13} X_1 X_3 + \beta_{23} X_2 X_3$$

$$+ \beta_{123} X_1 X_2 X_3 + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$$

Coding:

$X_1 = -1$ (T 29°C) $X_1 = +1$ (T 55°C)

$X_2 = -1$ (C 10mg/L) $X_2 = +1$ (C 1200mg/L)

$X_3 = -1$ (pH 2) $X_3 = +1$ (pH 6)

$2^3 = 8$ unique factor combinations

- 2 runs $\rightarrow$ $2 \cdot 2^3 = 16$ observations (n)
- All interactions : $p = 8$ (length $\vec{\beta}$)

* Do the following in combination with slides *

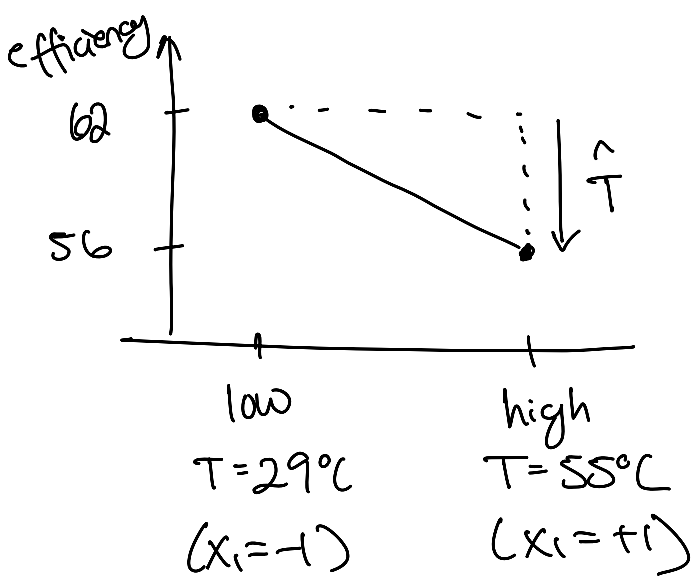
- Manual calculation of effects

$$\hat{T} = \underset{T \text{ high level}}{\text{mean } y,} - \underset{T \text{ low level}}{\text{mean } y,}$$

$$= \boxed{} - \boxed{}$$

$$= 56.5375 - 62.2 = -5.6625$$

interpret:



lower efficiency at high temperatures

Slope $= \frac{1}{2}(-5.6625)$

$= -2.83$

(similarly for $\hat{C}$, $\hat{pH}$)

$$\hat{TC} = \frac{1}{2} \cdot \text{main effect est. } T,\ \underline{C \text{ high level}} - \frac{1}{2} \cdot \text{main effect est. } T,\ \underline{C \text{ low level}}$$

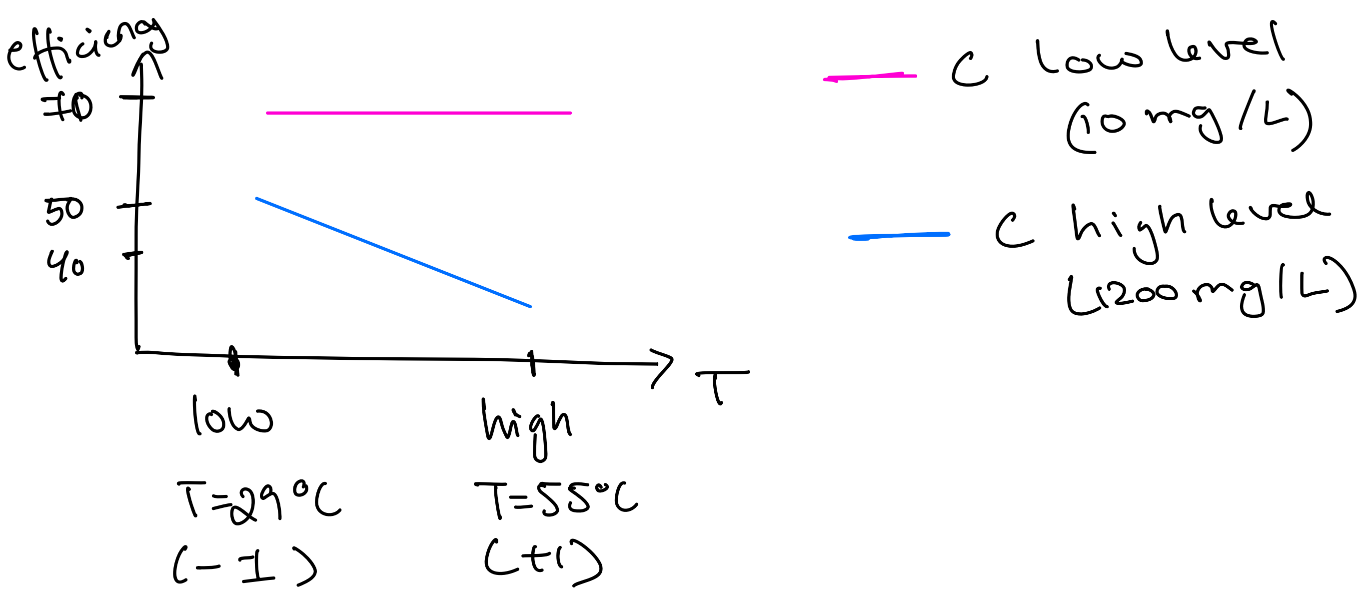
$$= \frac{1}{2}\left(\text{mean}\ \boxed{} - \text{mean}\ \boxed{}\right) - \frac{1}{2}\left(\boxed{} - \boxed{}\right)$$

$$= \frac{1}{2}(41.7 - 54.7) - \frac{1}{2}(71.375 - 69.7)$$

$$= \frac{1}{2}(-13) - \frac{1}{2}(1.675)$$

$$= -7.3375$$

interpret:



slope $= \frac{1}{2}(1.675)$

slope $= \frac{1}{2}(-13)$

} slope difference $= \hat{TC}$.

Note: "no" effect of temperature
when concentration at low level.

Which effects are significant?
Use $\alpha_{loc} = 0.05$ (no mult. test. correction in paper)

T, C, pH, TC, TpH, CpH (not TCpH)
$\ast$ $\ast$

$\ast$ would not reach $0.05/7 \approx 0.007$ (Bonferroni)

Example 2 : 2^3 factorial (1 run)

recall: length $\beta = 8$

$n = 1 \cdot 2^3 = 8$

no d.o.f to estimate σ^2 → no info regarding uncertainty.

e.g. estimate μ, σ^2 in $N(\mu, \sigma^2)$ from 1 observation

e.g. estimate $\beta_0, \beta_1, \sigma^2$ in $N(\beta_0 + \beta_1 x, \sigma^2)$ from 1 pair (x, y).

Solution 1: Do another run (as in ex. 1)

Solution 2: Lenth's method

recall $\hat{\beta} \sim N(\beta, \sigma^2/n I)$ independent, same variance

$H_0: \vec{\beta} = 0$

use: $Z \sim N(0, \tau^2)$ then $1.5 \text{ median} |Z| \approx \tau$

H_0: all $\beta = 0$ → we have 8 realizations of Z → estimate median $|Z|$ → estimate: $\frac{\sigma^2}{n}$. $\hat{\tau}$

T-test $\beta_j = 0$: $T_j = \frac{\hat{\beta}_j}{\hat{\tau}}$

Solution 3 : 'Sacrifice' some interaktions

full modell:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$
$$+ \beta_{12} x_1 x_2 + \underset{?}{\beta_{13} x_1 x_3} + \underset{?}{\beta_{12} x_1 x_3} + \underset{?}{\beta_{123} x_1 x_2 x_3} + \varepsilon$$

Note : especially useful in 2^4 and higher

Example 3: Blocking

2^2 experiment

Design matrix

$$X = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & A & B & AB \end{array}$$
$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

2 combinations on each computer, how to split

Try: Computer 1

A	B	AB
-1	-1	1
-1	1	-1

Computer 2

A	B	AB
1	-1	-1
1	1	1

problem: factor A is <u>confounded</u> by computer

Solution: Use an <u>interaction</u> as 'blocking factor'

Computer 1

A	B	AB
-1	-1	1
1	1	1

Computer 2

A	B	AB
-1	1	-1
1	-1	-1

Now AB is confounded by computer, but we were not interested in AB.

More complex: 2^3 with 4 blocks

A B C AB AC BC ABC

need two blocking factors

- try ABC and BC.

  NB: $ABC \cdot BC = ABBCC = A11 = A$

  $\rightarrow$ A is also confounded! (implicitly...)

- solution AB, AC

  Note $ABAC = AABC = BC$ also confounded.