Principal components

Principal components analysis is a method for re-expressing multivariate data. It allows the researcher to reorient the data so that the first few dimensions account for as much of the variation in the original variables as possible. Most often it is used for dimension reduction and data interpretation.

Let $\boldsymbol{X}' = (X_1, X_2, \dots, X_p)$ and let $\boldsymbol{\Sigma} = Cov(\boldsymbol{X})$. A linear combination of $\boldsymbol{X}$ is given by $Y = \boldsymbol{l}'\boldsymbol{X}$ and $Var(Y) = \boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l}$. For two linear combinations $Y_i = \boldsymbol{l}_i'\boldsymbol{X}$ and $Y_k = \boldsymbol{l}_k'\boldsymbol{X}$, we have that $Cov(Y_i, Y_k) = \boldsymbol{l}_i'\boldsymbol{\Sigma}\boldsymbol{l}_k$.

Principal components are uncorrelated linear combinations of $\boldsymbol{X}$ constructed such that

1. The first one $\underset{l_1'l_1=1}{max\,imizes}\ Var(\boldsymbol{l}_1'\boldsymbol{X})$
2. The second one $\underset{l_2'l_2=1}{max\,imizes}\ Var(\boldsymbol{l}_2'\boldsymbol{X})$ under the constraint $\boldsymbol{l}_1'\boldsymbol{\Sigma}\boldsymbol{l}_2 = 0$
3. The third one $\underset{l_3'l_3=1}{max\,imizes}\ Var(\boldsymbol{l}_3'\boldsymbol{X})$ under the constraints $\boldsymbol{l}_i'\boldsymbol{\Sigma}\boldsymbol{l}_3 = 0,\ \mathrm{i} < 3$

and so on.

<u>RP1</u>. *Let $\boldsymbol{X}_{p\times 1}$ has covariance matrix $\boldsymbol{\Sigma}$ with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0$ and eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_p$. The i-th principal component is given by $Y_i = \boldsymbol{e}_i'\boldsymbol{X},\ i = 1, 2, \dots, p,\ Var(Y_i) = \lambda_i$ and $Cov(Y_i, Y_k) = 0,\ i \ne k$.*

Proof

$$\boldsymbol{Y} = \boldsymbol{P}'\boldsymbol{X} \Rightarrow \boldsymbol{Cov}(\boldsymbol{Y}) = \boldsymbol{P}'\boldsymbol{\Sigma}\boldsymbol{P} = \boldsymbol{P}'\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}'\boldsymbol{P} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{bmatrix}$$

Further

$$\underset{l'l=1}{max}\ Var(\boldsymbol{l}'\boldsymbol{X}) = \underset{l'l=1}{max}\ Var\left(\frac{\boldsymbol{l}'\boldsymbol{X}}{\boldsymbol{l}'\boldsymbol{l}}\right) = \underset{l'l=1}{max}\frac{\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l}}{\boldsymbol{l}'\boldsymbol{l}}$$

$$= \underset{l'l=1}{max}\frac{\boldsymbol{l}'\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}'\boldsymbol{l}}{\boldsymbol{l}'\boldsymbol{P}\boldsymbol{P}'\boldsymbol{l}} = \underset{l'l=1}{max}\frac{\boldsymbol{y}'\boldsymbol{\Lambda}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{y}} = \underset{l'l=1}{max}\frac{\sum_{i=1}^{p}\lambda_i y_i^2}{\sum_{i=1}^{p} y_i^2} \le \lambda_1$$

But choosing $\boldsymbol{l} = \boldsymbol{e}_1$ gives $\frac{\boldsymbol{e}_1'\boldsymbol{\Sigma}\boldsymbol{e}_1}{\boldsymbol{e}_1'\boldsymbol{e}_1} = \lambda_1$ which implies $\underset{l'l=1}{max}\ Var(\boldsymbol{l}'\boldsymbol{X}) = \lambda_1$.

Also $\max\limits_{l \perp e_1,\ldots,e_k, l'l=1} \frac{l'\Sigma l}{l'l} = \max\limits_{l \perp e_1,\ldots,e_k} \frac{l' P\Lambda P' l}{l' PP' l} = \frac{\sum_{i=k+1}^{p} \lambda_i y_i^2}{\sum_{i=k+1}^{p} y_i^2} \le \lambda_{k+1}$

but $\frac{e_{k+1}^t \Sigma e_{k+1}}{e_{k+1}^t e_{k+1}} = \frac{e_{k+1}^t \lambda_{k+1} e_{k+1}}{e_{k+1}^t e_{k+1}} = \lambda_{k+1}$.

RP2. *Let $X_{p\times 1}$ has covariance matrix $Cov(X) = \Sigma$. Let $Y_i = e_i^t X, i = 1,2,\ldots,p$ be the principalcomponents. Then $tr(\Sigma) = \sum_{i=1}^{p} \sigma_{ii} = \sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} Var(Y_i)$.*

Proof. $tr(\Sigma) = tr(P\Lambda P^t) = tr(P^t P\Lambda) = tr(\Lambda)$□

Definition

Total population variance is defined as $\sum_{i=1}^{p} \sigma_{ii}$. Hence $\frac{\lambda_k}{\sum_{i=1}^{p} \lambda_i}$ explains how much of the total population variance that is explained by the k-th principal component.

The figures on the next page are trying to illustrate what happens when we do a principal component analysis. Z_1, Z_2 and Z_3 are the principal components. The figures are taken from the book: Analyzing Multivariate Data by J. Latin, J. C. Carroll and P. E. Green.

FIGURE 4.3
Stylized three-dimensional view of shape of distribution of X_1, X_2, and X_3. Shadows represent shapes in two dimensions

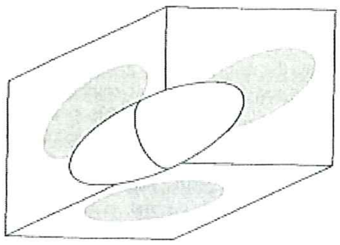


FIGURE 4.4
Three-dimensional scatter plot of actual values of X_1, X_2, and X_3

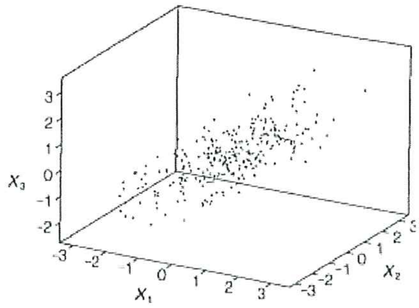


FIGURE 4.5
Pairwise scatter plots of X_1 versus X_2, X_1 versus X_3, and X_2 versus X_3

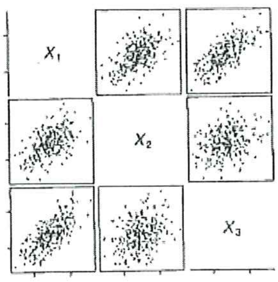


FIGURE 4.6
Stylized three-dimensional view identifying first principal component

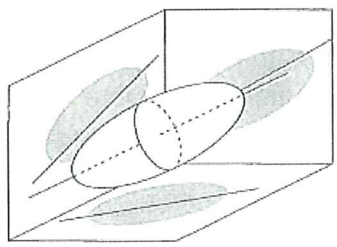


FIGURE 4.10
Stylized three-dimensional view of shape of distribution of Z_1, Z_2, and Z_3

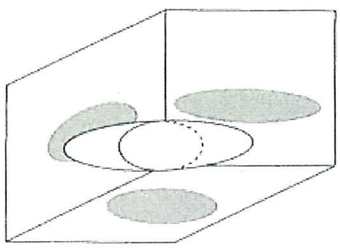
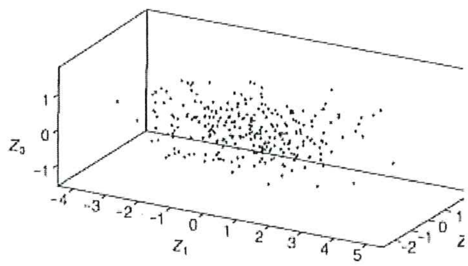


FIGURE 4.11
Three-dimensional scatter plot of actual values of Z_1, Z_2, and Z_3



RP3. *Let* $Y_i = e_i' X, i = 1,2,\dots,p$ *be the principal components of* $\boldsymbol{\Sigma}$. *Then*

$$\rho_{Y_i,X_k} = \frac{e_{ik}\sqrt{\lambda_i}}{\sqrt{\sigma_{kk}}},\quad i,k = 1,2,\dots,p.$$

Proof. Let $X_k = [0,\dots,0,1,0,\dots,1]X = l_k' X$. Then
$Cov(X_k, Y_i) = E\left(l_k'(X-u)(X-u)' e_i\right) = l_k'\Sigma e_i = l_k'\lambda_i e_i = \lambda_i e_{ik}$. Since the variance of Y_i equals λ_i, we have $\rho_{Y_i,X_k} = \dfrac{\lambda_i e_{ik}}{\sqrt{\lambda_i}\sqrt{\sigma_{kk}}}$. □

The correlations between the principal components and the original variables are called principal component loadings. These may also be informative in explaining how much of the variation in an original random variable that is explained by the respective principal components. In simple linear regression we have:

$$SS_E = \sum_{i=1}^{n}\left(y_i - a - bx_i\right)^2 = \sum_{i=1}^{n}\left((y_i-\bar{y}) - b(x_i-\bar{x})\right)^2$$
$$= \sum_{i=1}^{n}(y_i-\bar{y})^2 + b^2\sum_{i=1}^{n}(x_i-\bar{x})^2 - 2b\sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})$$

Recalling that $b = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$ we obtain

$$SS_E = SS_T - \frac{\left(\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})\right)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = SS_T - SS_R.$$

Hence $R^2 = \dfrac{SS_R}{SS_T} = \dfrac{\left(\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})\right)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2} = \hat{\rho}_{YX}^2$

Let us assume p=3 for instance such that the $\hat{\rho}_{Y_1X_1} = \dfrac{\sqrt{\lambda_1}e_{11}}{\sqrt{\sigma_{11}}} = 0.9279$,

$\hat{\rho}_{Y_2X_1} = \dfrac{\sqrt{\lambda_2}e_{21}}{\sqrt{\sigma_{11}}} = -0.0798$ and $\hat{\rho}_{Y_3X_1} = \dfrac{\sqrt{\lambda_3}e_{31}}{\sqrt{\sigma_{11}}} = -0.3641$. Then since principal components are orthogonal we will know that in a regression of X_1 on Y_1, Y_2 and Y_3, Y_1 explains 0.93^2 $= 0.86$ or 86% percent of the variation in X_1, Y_2 explains $0.08^2 = 0.0064 = 0.64\%$ and Y_3 the rest.

Let $Z_i = \frac{X_i - \mu}{\sqrt{\sigma_{ii}}}$, $i = 1, 2, \ldots, p$ or

$$Z_{p\times 1} = \begin{bmatrix} \sigma_{11}^{-1/2} & 0 & \cdots & 0 \\ 0 & \sigma_{22}^{-1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{pp}^{-1/2} \end{bmatrix} \begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_p - \mu_p \end{bmatrix} = V^{-\frac{1}{2}}(X - \mu)$$

$\Rightarrow Cov(Z) = V^{-1/2}\Sigma V^{-1/2} = \rho$ i.e. the correlation matrix. Hence the principal components of Z are found by means of the eigenvectors of ρ.

RP4. *Let $Z_{p\times 1}$ be standardized and let $Cov(Z) = \rho$. Let $Y_i = e_i' Z$, $i = 1, 2, \ldots, p$ be the principal components of $Z_{p\times 1}$. Then $\sum_{i=1}^{p} Var(Y_i) = \sum_{i=1}^{p} Var(Z_i) = p$.*

Proof. Follows from RP2.

Principal components derived from Σ are different from those obtained from ρ.
A random variable with much higher variance then the others will dominate the first principal components when principal components are derived from the covariance matrix. The normal procedure is to standardize the variables if they are measured on scales with different ranges or if their units are not commensurable. On the other side when making questionnaires for instance, it may happen that some questions turn out to be less informative in the sense that almost all persons answers the same. Standardizing these variables may add to much values to these answers and a non standardized analysis is to be preferred.

In practice we normally work with the estimated covariance matrix or correlation matrix.
Also it is common to center the variables such that we work with $Y_i = e_i'(X - \mu)$ or in practise with $\hat{y}_i = \hat{e}_i'(x - \bar{x}), i = 1, 2, \ldots, p$. Centering does not affect the calculation of eigenvectors or eigenvalues.

In this example the athletic records for 55 countries were investigated for the eight running distances: Y_1=100m, Y_2=200m, Y_3=400m, Y_4=800m, Y_5=1500, Y_6=5000m, Y_7=10000m and Y_8=marathon.

These are given on the next page.

Country	100m	200m	400m	800m	1500m	5000m	10000m	Marathon
Argentina	10.39	20.81	46.84	1.81	3.70	14.04	29.36	137.72
Australia	10.31	20.06	44.84	1.74	3.57	13.28	27.66	128.30
Austria	10.44	20.81	46.82	1.79	3.60	13.28	27.72	135.90
Belgium	10.34	20.68	45.04	1.73	3.60	13.22	27.45	129.95
Bermuda	10.28	20.58	45.91	1.80	3.75	14.68	30.55	146.62
Brazil	10.22	20.43	45.21	1.73	3.66	13.62	28.82	133.13
Burma	10.64	21.52	48.30	1.80	3.85	14.45	30.28	139.95
Canada	10.17	20.22	45.68	1.76	3.63	13.55	28.09	130.15
Chile	10.34	20.80	46.20	1.79	3.71	13.61	29.30	134.03
China	10.51	21.04	47.30	1.81	3.73	13.90	29.13	133.53
Colombia	10.43	21.05	46.10	1.82	3.74	13.49	27.88	131.35
Cook Is	12.18	23.20	52.94	2.02	4.24	16.70	35.38	164.70
Costa Rica	10.94	21.90	48.66	1.87	3.84	14.03	28.81	136.58
Czech	10.35	20.65	45.64	1.76	3.58	13.42	28.19	134.32
Denmark	10.56	20.52	45.89	1.78	3.61	13.50	28.11	130.78
Dom Rep	10.14	20.65	46.80	1.82	3.82	14.91	31.45	154.12
Finland	10.43	20.69	45.49	1.74	3.61	13.27	27.52	130.87
France	10.11	20.38	45.28	1.73	3.57	13.34	27.97	132.30
GDR	10.12	20.33	44.87	1.73	3.56	13.17	27.42	129.92
FRG	10.16	20.37	44.50	1.73	3.53	13.21	27.61	132.23
GB	10.11	20.21	44.93	1.70	3.51	13.01	27.51	129.13
Greece	10.22	20.71	46.56	1.78	3.64	14.59	28.45	134.60
Guatemala	10.98	21.82	48.40	1.89	3.80	14.16	30.11	139.33
Hungary	10.26	20.62	46.02	1.77	3.62	13.49	28.44	132.58
India	10.60	21.42	45.73	1.76	3.73	13.77	28.81	131.98
Indonesia	10.59	21.49	47.80	1.84	3.92	14.73	30.79	148.83
Ireland	10.61	20.96	46.30	1.79	3.56	13.32	27.81	132.35
Israel	10.71	21.00	47.80	1.77	3.72	13.68	28.93	137.55
Italy	10.01	19.72	45.26	1.73	3.60	13.23	27.52	131.08
Japan	10.34	20.81	45.86	1.79	3.64	13.41	27.72	128.63
Kenya	10.48	20.66	44.92	1.73	3.55	13.10	27.80	129.75
Korea	10.34	20.89	46.90	1.79	3.77	13.96	29.23	136.25
P Korea	10.91	21.94	47.30	1.85	3.77	14.13	29.67	130.87
Luxemburg	10.35	20.77	47.40	1.82	3.67	13.64	29.08	141.27
Malaysia	10.40	20.92	46.30	1.82	3.80	14.64	31.01	154.10
Mauritus	11.19	22.45	47.70	1.88	3.83	15.06	31.77	152.23
Mexico	10.42	21.30	46.10	1.80	3.65	13.46	27.95	129.20
Netherlands	10.52	20.95	45.10	1.74	3.62	13.36	27.61	129.02
NZ	10.51	20.88	46.10	1.74	3.54	13.21	27.70	128.98
Norway	10.55	21.16	46.71	1.76	3.62	13.34	27.69	131.48
Png	10.96	21.78	47.90	1.90	4.01	14.72	31.36	148.22
Philippines	10.78	21.64	46.24	1.81	3.83	14.74	30.64	145.27
Poland	10.16	20.24	45.36	1.76	3.60	13.29	27.89	131.58
Portugal	10.53	21.17	46.70	1.79	3.62	13.13	27.38	128.65
Rumania	10.41	20.98	45.87	1.76	3.64	13.25	27.67	132.50
Singapore	10.38	21.28	47.40	1.88	3.89	15.11	31.32	157.77
Spain	10.42	20.77	45.98	1.76	3.55	13.31	27.73	131.57
Sweden	10.25	20.61	45.63	1.77	3.61	13.29	27.94	130.63
Switzerland	10.37	20.45	45.78	1.78	3.55	13.22	27.91	131.20
Tapai	10.59	21.29	46.80	1.79	3.77	14.07	30.07	139.27
Thailand	10.39	21.09	47.91	1.83	3.84	15.23	32.56	149.90
Turkey	10.71	21.43	47.60	1.79	3.67	13.56	28.58	131.50
USA	9.93	19.75	43.88	1.73	3.53	13.20	27.43	128.22
USSR	10.07	20.00	44.60	1.75	3.59	13.20	27.53	130.56

```
W Samoa       10.82   21.86 49.00   2.02   4.24    16.28  34.71    161.83
```

The correlation matrix is given by:

```
> cora=cor(athletic)
> cora
              X100m     X200m     X400m     X800m    X1500m    X5000m    X10000m
X100m     1.0000000 0.9220672 0.8400364 0.7549836 0.6992392 0.6190243 0.6325579
X200m     0.9220672 1.0000000 0.8505850 0.8065120 0.7750567 0.6957027 0.6956162
X400m     0.8400364 0.8505850 1.0000000 0.8703090 0.8353696 0.7796495 0.7839919
X800m     0.7549836 0.8065120 0.8703090 1.0000000 0.9180442 0.8637763 0.8659091
X1500m    0.6992392 0.7750567 0.8353696 0.9180442 1.0000000 0.9282907 0.9335240
X5000m    0.6190243 0.6957027 0.7796495 0.8637763 0.9282907 1.0000000 0.9736935
X10000m   0.6325579 0.6956162 0.7839919 0.8659091 0.9335240 0.9736935 1.0000000
Marathon  0.5191498 0.5962022 0.7050590 0.8064734 0.8655467 0.9325996 0.9426427
           Marathon
X100m     0.5191498
X200m     0.5962022
X400m     0.7050590
X800m     0.8064734
X1500m    0.8655467
X5000m    0.9325996
X10000m   0.9426427
Marathon  1.0000000
```

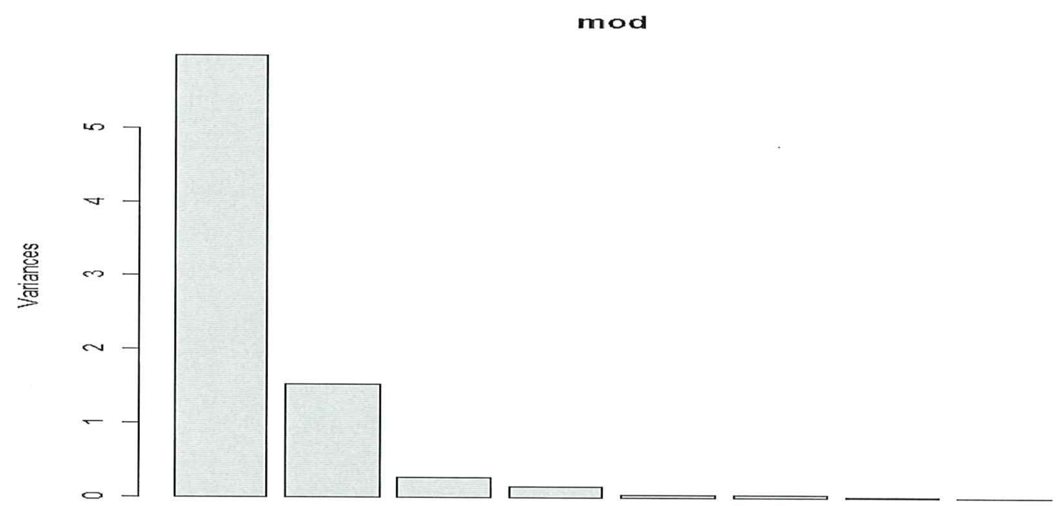
And the principal components can be found by:

```
> mod=prcomp(cora, scale=TRUE)
> mod
Standard deviations:
[1] 2.446849e+00 1.235417e+00 5.208693e-01 3.849336e-01 1.842070e-01
[6] 1.661118e-01 7.530036e-02 2.941575e-17
Rotation:
                PC1         PC2         PC3         PC4         PC5         PC6
X100m    -0.3972462  0.13787010  0.17041822  0.16466694  0.44034632  0.52509540
X200m    -0.3750790  0.21241656  0.39437898  0.46616656 -0.63857445 -0.13981278
X400m    -0.2297833  0.58511062 -0.76730254  0.09822395 -0.06088915 -0.05155211
X800m     0.1864080  0.67796967  0.38688732 -0.57572338 -0.09261032  0.12271880
X1500m    0.3532475  0.35222297  0.21735415  0.49781875  0.49448212 -0.45444407
X5000m    0.4023910  0.05994106 -0.05438611  0.29761455 -0.25166613  0.48901051
X10000m   0.4039110  0.04790494 -0.06768067  0.26703365 -0.05953095  0.43667849
Marathon  0.4051130 -0.06264907 -0.14901850 -0.08223922 -0.27320558 -0.22510196
                  PC7          PC8
X100m     0.143599156  0.526097865
X200m     0.114495859  0.030243952
X400m     0.004285942 -0.007783854
X800m     0.005125173 -0.028409797
X1500m   -0.070745597  0.007207619
X5000m   -0.653891995  0.113641827
X10000m   0.670796424 -0.338339269
Marathon  0.289271190  0.770713149
> summary(mod)
```

```
Importance of components:
                          PC1    PC2     PC3     PC4     PC5     PC6     PC7
Standard deviation     2.4468 1.2354 0.52087 0.38493 0.18421 0.16611 0.07530
Proportion of Variance 0.7484 0.1908 0.03391 0.01852 0.00424 0.00345 0.00071
Cumulative Proportion  0.7484 0.9392 0.97308 0.99160 0.99584 0.99929 1.00000
                             PC8
Standard deviation     2.942e-17
Proportion of Variance 0.000e+00
Cumulative Proportion  1.000e+00
```

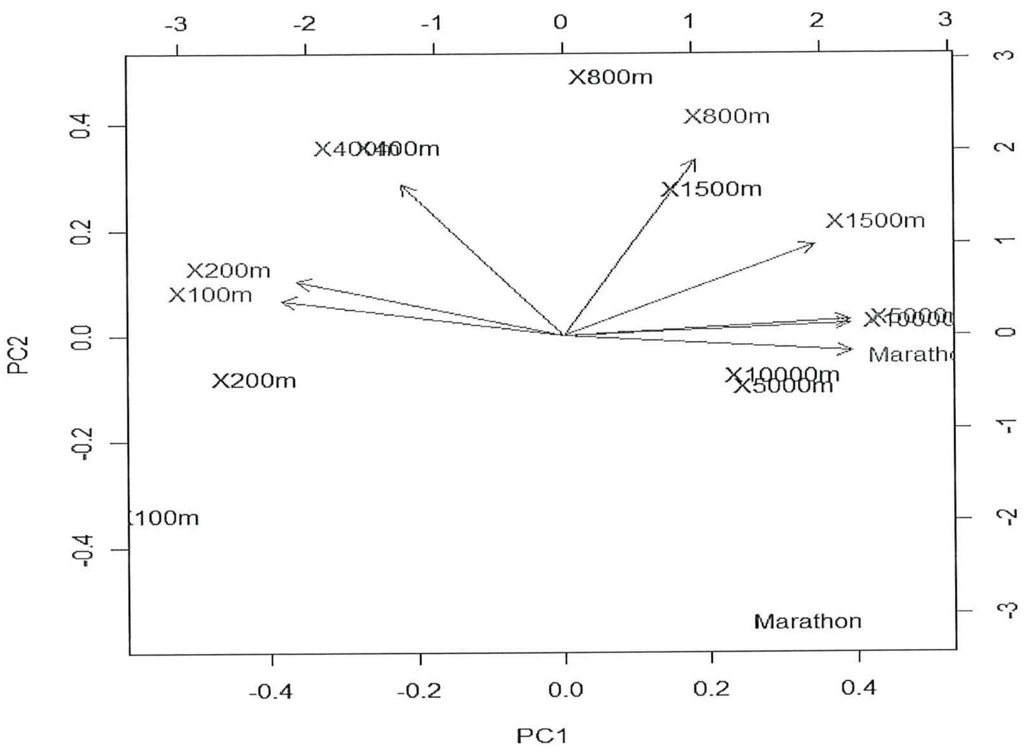
The standard deviations are the square root of the eigenvalues. In order to determine how many principal components we should use, we can use a scree plot. In a scree plot we try to find the "elbow" and go one step to the left. In this case most people would place the elbow at three, which indicates that two components should be used. Sometimes we prefer (depends on the dimension) to leave out components whose corresponding eigenvalues are less than one.

```
> plot(mod)
```

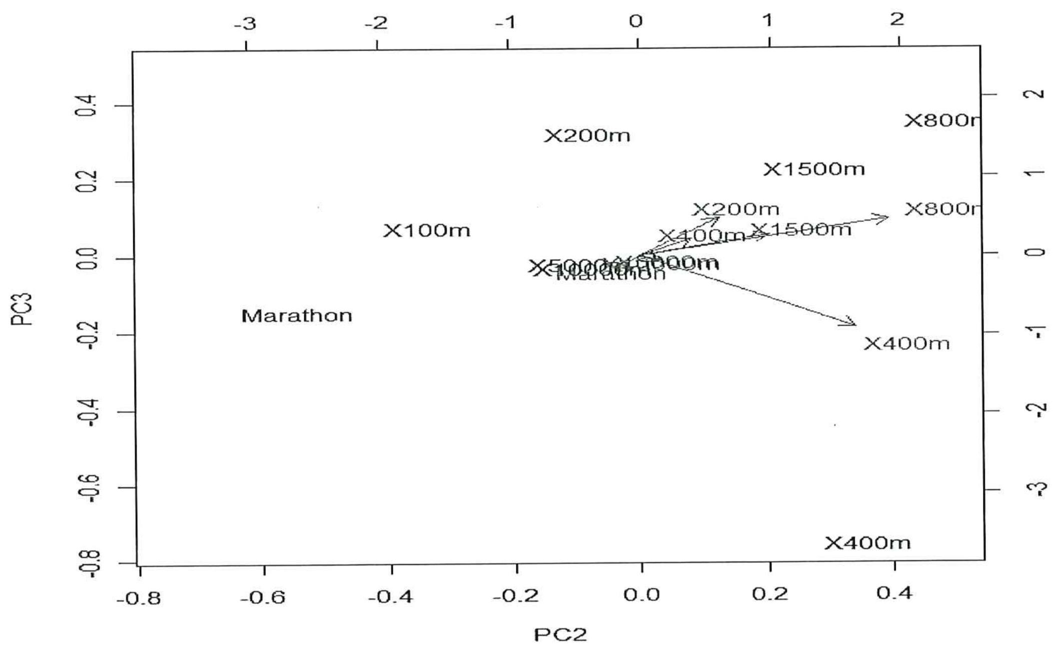


In this case two principal components explain almost 93.9 % of the variation in the data. When the original data are standardized and the principal component analysis is built on the correlation matrix, then since in this case $\sigma_{kk} = 1$, the principal component loadings between principal component *i* and the standardized variables are directly proportional to eigenvector *i*. In a biplot the second eigenvector is plotted against the first. Note the grouping. First the short distances in one side of the plot and the "middle" and "long" distances on the other. Than the 100m and 200m come out almost identical. The same with 5000m and 10000m.

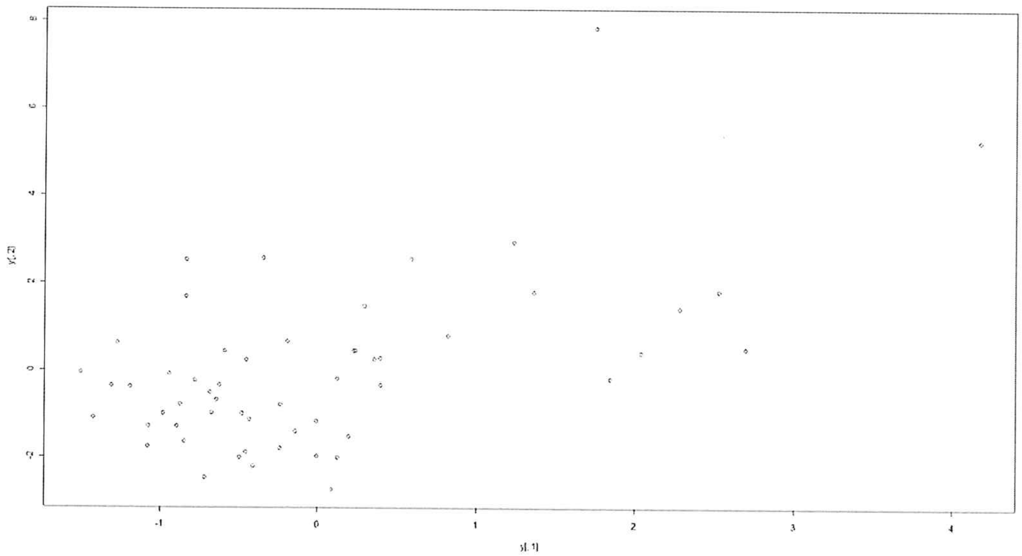
```
> biplot(mod)
```



`> biplot(mod, choices=2:3)` (Shows a command for plotting other eigenvectors against each other)



The essential conclusion is that the dimension of the data can be reduced from eight to two if we are using the principal components. If $\boldsymbol{Z}$ refers to the standardized variables, we may compute $Y_1 = \boldsymbol{e}_1^t \boldsymbol{Z}$ and $Y_2 = \boldsymbol{e}_2^t \boldsymbol{Z}$ for each of the 55 countries. These are called the principal component scores. A plot of Y_2 against Y_1 is shown below. The one country with the very large y_1 value is West Somoa. The one with the large y_2 value is Cook Island. The y_1 and y_2 values for the other countries are given below. Norway is located far to the left.



```
          Y1                Y2

> y
              pr1          pr2
 [1,]  0.385192592  0.28503390
 [2,] -0.457089967 -1.87300961
 [3,] -0.782314864 -0.23028309
 [4,] -0.848270680 -1.63712979
 [5,]  1.848473501 -0.18202901
 [6,]  0.190180804 -1.51793205
 [7,]  0.286762814  1.47531399
 [8,] -0.013158584 -1.17051891
 [9,]  0.118437940 -0.19393456
[10,] -0.192427467  0.67227448
[11,] -0.454161930  0.24201765
[12,]  1.744860851  7.85441051
[13,] -0.834950125  2.53768755
[14,] -0.433457951 -1.12295648
[15,] -0.640194968 -0.67412673
[16,]  2.693074571  0.50448325
[17,] -0.891631482 -1.28937295
[18,] -0.238614603 -1.79060799
[19,] -0.493866380 -2.00153321
[20,] -0.409494926 -2.19674123
[21,] -0.717598742 -2.45728053
[22,]  0.389832310 -0.34366250
[23,] -0.348388575  2.57087694
[24,] -0.237912300 -0.79496306
```

```
[25,] -0.625804583 -0.33808305
[26,]  1.357295648  1.80280003
[27,] -1.188094833 -0.38375902
[28,] -0.589974282  0.44081660
[29,]  0.119481024 -1.99908463
[30,] -0.685631461 -0.50548177
[31,] -1.080136418 -1.74811591
[32,]  0.352196816  0.26149954
[33,] -0.840509874  1.69120138
[34,]  0.221397386  0.45817825
[35,]  2.038497946  0.41753855
[36,]  0.584244297  2.56416052
[37,] -0.944175885 -0.08153226
[38,] -1.077992705 -1.28085260
[39,] -1.417103372 -1.09529428
[40,] -1.306246315 -0.35655230
[41,]  1.230412222  2.94418279
[42,]  0.818010225  0.79857831
[43,] -0.145030002 -1.39871491
[44,] -1.499489332 -0.05042235
[45,] -0.874395411 -0.77721206
[46,]  2.528855677  1.83028642
[47,] -0.981417326 -0.98889651
[48,] -0.481866739 -0.99640779
[49,] -0.670833183 -0.98057707
[50,]  0.228648266  0.46102597
[51,]  2.283667797  1.42717981
[52,] -1.268527428  0.62488720
[53,]  0.084442906 -2.71812022
[54,] -0.009954145 -1.95665878
[55,]  4.176751247  5.26741357
```

Some commands used to obtain Y_1 and Y_2.

```
> atl=read.table("F:/athletic.txt",header=T)
> atlc=scale(atl) (standardiserer observasjonane)
> atlcc=cbind(atlc[,1],atlc[,2],atlc[,3],atlc[,4],atlc[,5],atlc[,6],atlc[,7],atlc[,8])
> pr1=c(-0.397,-0.375,-0.223,0.186,0.353,0.402,0.404,0.405)
> pr2=c(0.138,0.212,0.585,0.678,0.352,0.060,0.048,-0.063)
> prr=cbind(pr1,pr2)
> y=atlcc%*%prr
```